Vector Spaces

Important Quetions

① Define vector Space, Subspace with examples ST $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} s, t \in R \right\}$ is subspace of R^3

② (i) Show that Intersection of two Subspaces is a Subspace

(ii) if H and K are two subspaces of a vector V show that H+K is also a subspace of V.

③ State and prove spanning Set theorem

④ Find the Spanning set for Nullspace of matrix $A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$

⑤ Define Basis and find Basis for Null A and col A where $A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$

⑥ find basis and dimension of $H = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} s, t \in R \right\}$

(or)

Define dimension and find Dimension of Subspace $H = \left\{ \begin{bmatrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{bmatrix} : a, b, c, d \in R \right\}$

⑦ Define Linearly Independent, linearly dependent sets and find dimension of Subspace. Spanned by vectors $\begin{bmatrix} +1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix} \begin{bmatrix} -3 \\ 5 \\ 5 \end{bmatrix}$

Vectorspace, Subspace

Spanning set theorem

Nulspace, column space

Vectorspaces

Dimension

Linear Transformations

Co-ordinate systems

Linearly dependent
Linearly Independent sets

Bases

(9) Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$ $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$ and $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$

Determine if $\{v_1, v_2, v_3\}$ is a basis for R^3

(8) The set $\beta = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis for P_2. find co-ordinate vector of $P(t) = 1+4t+7t^2$ relative to β

① **Vector space** :- A set V of vectors with 2 operations Addition and multiplication by Scalars.

→ V cannot be a empty set

If $\bar{u}$, $\bar{v}$, and $\bar{w}$ are vectors in V and lets c and d are 2 Scalars then,

Addition	Multiplication
① $\bar{u}+\bar{v} \in V$, $\bar{u}+\bar{v}=\bar{v}+\bar{u}$	① $\bar{u} \in V$ c is Scalar $c\bar{u} \in V$
② $(\bar{u}+\bar{v})+\bar{w} = \bar{u}+(\bar{v}+\bar{w})$	② $c(\bar{u}+\bar{v}) = c\bar{u}+c\bar{v}$
③ Zero vector $\bar{0}$ in V $\bar{u}+\bar{0}=\bar{u}$	③ $(c+d)\bar{u} = c\bar{u}+d\bar{u}$
④ for each u in V there exists $-u$ in V $\bar{u}+(-\bar{u})=\bar{0}$	④ $c(d\bar{u}) = (cd)\bar{u}$
	⑤ $1\bar{u} = \bar{u}$

② **Subspace** :-

A subset H of a vector space V that is itself a vector space under same operations.

(i) The zero vector of V is in H

(ii) for $\bar{u}$ and $\bar{v}$ in H, Sum $\bar{u}+\bar{v}$ is in H

(iii) for each each vector $\bar{u}$ in H and each Scalar c $c\bar{u}$ is in H

H is closed under addition

H is closed under multiplication by Scalars

(1) ST $H = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$ is subspace of R^3

$$H = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$$

$$H = s\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$sv_1 + tv_2 + 0v_3$

$H = \text{span}\{v_1, v_2, v_3\}$

$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ & $v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

v_1, v_2 are in vectorspace R^3

so H is subspace of R^4

(ii) $H = \begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix}$

$$H = a\begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$H = av_1 + bv_2$

$H = \text{Span}\{v_1, v_2\}$

$v_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}$

v_1, v_2 are in vectorspace R^4

H is subspace R^4

(3) V_1 and V_2 in vector space V, let $H = \text{span}\{v_1, v_2\}$ show that H is subspace of V

$H = \text{span}\{v_1, v_2\}$.

V_1 and V_2 in vector space V

~~Two~~ to show that H is subspace of V

- $\bar{0}$ vector in H
- closed addition
- closed scalar multiplication

(i) The zero vector is in H, ~~0~~ v_1

$\bar{0} \in H$.

$0 = 0v_1 + 0v_2$.

let $\bar{u}$ and $\bar{w}$ in H c_1, c_2 and d_1, d_2 are scalars

then $\bar{u} = c_1v_1 + d_1v_2$ $\bar{w} = c_2v_1 + d_2v_2$.

$$\bar{u} + \bar{w} = c_1v_1 + d_1v_2 + c_2v_1 + d_2v_2$$

$$\bar{u} + \bar{w} = (c_1 + c_2)v_1 + (d_1 + d_2)v_2$$

$$\bar{u} + \bar{w} = cv_1 + dv_2 = \text{span}\{v_1, v_2\}$$

$\therefore$ $\bar{u} + \bar{w}$ is in H $\Rightarrow$ $\bar{H}$ is closed under vector addition

To show that H is closed under Scalar multiplic

c is any Scalar $\bar{u}$ is vector $\Rightarrow$ $c\bar{u} \in V$

if $\bar{u}$ is in H let c is any Scalar

$\bar{u} = c_1 v_1 + c_2 v_2$

$c\bar{u} = c[c_1 v_1 + c_2 v_2]$

$= cc_1 v_1 + cc_2 v_2$

$= (cc_1) v_1 + (cc_2) v_2$

$c\bar{u} = \text{span}\{v_1, v_2\}$

$\therefore$ $c\bar{u}$ is in H

$\therefore$ H is closed under multiplication by Scalars

1 2 If H and K are Subspaces of a vector space V then show that H intersection K is also a subspace of V

RTP:- $H \cap K$ is subspace of V is wt we need to prove

$H \cap K \leftarrow$ zero vector
closed vector addition
closed multiplication by Scalars

proof :- V is vectorspace H and K subspaces V

$\Rightarrow$ if H is Subspace of $V \Rightarrow \bar{0} \in H$

K is subspace of $V \Rightarrow \bar{0} \in K$

So $\bar{0} \in H \cap K$

if $\bar{u}, \bar{w} \in H \cap K$

$\Rightarrow$ $\bar{u}$ and $\bar{w} \in H \Rightarrow \bar{u} + \bar{w} \in H$ [H is subspace]

$\bar{u}$ and $\bar{w} \in K \Rightarrow \bar{u} + \bar{w} \in K$ [K is subspace]

$\bar{u} + \bar{w} \in H \cap K$ if $\bar{u}, \bar{w} \in H \cap K$

$H \cap K$ is closed under addition

if $\bar{u} \in H \cap K$ and c is any Scalar

$\bar{u} \in H$ and $c\bar{u} \in H$

$\bar{u} \in K$ and $c\bar{u} \in K$

$\therefore$ $c\bar{u} \in H \cap K$ $\Rightarrow$ closed under Scalar multiplication

$H \cap K$ is a subspace of V when H and K are its subspaces.

9 (ii) If H and K are subspaces of a vector space V show that the sum $H+K$ is also a subspace of V

RTP:- $H+K$ is subspace ⟨ zero vector, addition, multiplication } closure

Proof:- given H and K are subspaces of V

$\Rightarrow \bar{0} \in H$, $\bar{0} K \Rightarrow \bar{0} + \bar{0} = \bar{0} \in H+K$

zero vector of $\bar{V}$ is in $H+K$.

$\Rightarrow$ if $\bar{u}$ and $\bar{w}$ are in $H+K$

$\bar{u} = h_1 + k_1$ and $w_2 = h_2 + k_2$ for $h_1, h_2 \in H$, $k_1, k_2 \in K$

$$\bar{u} + \bar{w} = h_1 + k_1 + h_2 + k_2$$
$$= (h_1 + h_2) + (k_1 + k_2)$$

Since $h_1 + h_2 \in H$ and $k_1 + k_2 \in K$.

$$\bar{u} + \bar{w} = (h_1 + h_2) + k_1 + k_2 \in H+K$$

$H+K$ is closed under vector addition.

if $\bar{u} = h_1 + k_1 \in H+K$ and c is any Scalar

$$c\bar{u} = c[h_1 + k_1] = ch_1 + ck_1 \in H+K$$

Since $ch_1 \in H$ and $ck_1 \in K$.

$\therefore$ $H+K$ is closed under Scalar multiplication

$\therefore$ $H+K$ is subspace of V

3

Spanning Set Theorem :-

3 Statement :- Let $S = \{v_1, v_2, \dots v_p\}$ is a set in V and Let $H = \text{Span}\{v_1, v_2, \dots v_p\}$

a) If one of vectors in S say, v_k is a linear combination of remaining vectors in S, then the set formed from S by removing v_k still spans H.

(b) If $H \neq \{0\}$ Some subset of S is a basis for H

Proof :-

let $S = \{v_1, v_2, \dots v_{p-1}, v_p\}$

v_p is a linear combination of vectors $v_1, v_2, \dots v_{p-1}$ then

$$v_p = a_1 v_1 + a_2 v_2 + a_3 v_3 + \dots + a_{p-1} v_{p-1}$$

where $a_1, a_2, a_3 \dots a_{p-1}$ are Scalars

let $x \in H \Rightarrow x \in \text{Span}\{v_1, v_2, \dots v_{p-1}, v_p\}$

$$X = c_1 v_1 + c_2 v_2 + \dots c_{p-1} v_{p-1} + c_p v_p$$

$$X = c_1 v_1 + c_2 v_2 + \dots c_{p-1} v_{p-1} + c_p [a_1 v_1 + a_2 v_2 \dots + a_{p-1} v_{p-1}]$$

$$x = (c_1 + c_p a_1) v_1 + (c_2 + c_p a_2) v_2 + \dots (c_{p-1} + c_p a_p) v_{p-1}$$

$\Rightarrow$ x = linear Combination of $v_1, v_2 \dots v_{p-1}$

$X = \text{Span}\{v_1, v_2, \dots v_{p-1}\}$ $x \in H$.

$\therefore$ The set $\{v_1, v_2, \dots v_{p-1}\}$ Spans H

from statement $H = \text{Span}\{v_1, v_2, \dots v_p\}$

(b) if $S = \{v_1, v_2, \dots v_p\}$ is linearly ~~de~~ independent

then S is basis for H.

otherwise if any vector in S dependends on others then it can be removed as we did in part (a.) and the remaining set still spanns a with linear independent vectors.

~~So it will be basis of H.~~

So as long as there are two/more dependent vectors we can repeat process until spanning set is linearly Independent and hence is a basis for H.

If spanning set reduced to one vector that vector will be non-zero because $H \neq \{0\}$

Linearly Independent :-

The set of vectors $\{v_1, v_2, v_3 \ldots v_p\}$ in vector space V said to be linear Independent if vector equation $c_1 v_1 + c_2 v_2 + \ldots + c_p v_p = 0$ has trival solution $c_1 = 0, c_2 = 0, \ldots c_p = 0$.

Linearly dependent :-

The set of $\{v_1, v_2, v_3 \ldots v_p\}$ is said to be linearly dependent if it has non trival solutions the vector equation $c_1 v_1 + c_2 v_2 + \ldots c_p v_p = \bar{0}$ and there exists sume scalars in $c_1, c_2 \ldots c_p$ i.e $c_1, c_2, \ldots c_p$ not all zeroes in

$$c_1 v_2 + c_2 v_2 + \ldots c_p v_p = \bar{0}$$

Theorem :- Set $\{v_1, v_2, \ldots v_p\}$ of 2 or more vectors with $v_1 \neq \bar{0}$ is linear dependent if and only if som v_j $(j>1)$ is linear combination of preceeding vectors, $v_1, v_2 \ldots v_{j-1}$

i.e $v_j = c_1 v_1 + c_2 v_2 + \ldots c_{j-1} v_{j-1} \Rightarrow \{v_1, v_2 \ldots v_j, \ldots v_p\}$ linearly dependent.

Basis :- H be a subspace of vectorspace V.

$B = \{b_1, b_2, \ldots b_p\}$ is a set of vectors in V

B is basis for H if.

(i) B is linearly Independent set.

(ii) Subspace spanned by B coincides with H

i.e $H = \text{Span}\{b_1, b_2, \ldots b_p\}$.

Example :- $H = c v_1 + d v_2 . \rightarrow v_2 \neq s v_1$

$H = \text{span}\{v_1, v_2\}$ $\longrightarrow$ Linear Independent Span

Set $\{v_1, v_2\}$ is basis of H

$v_1 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$ $v_2 = \begin{bmatrix} 2 \\ 12 \end{bmatrix} \Rightarrow v_2 = 2v_1$ is not possible

Nullspace and column space of a matrix

Null space of a matrix :-

The null space of a $m \times n$ matrix A written as Null A is solutions to homogeneous equation

$$\boxed{AX = 0}$$ Null space of $A_{m\times n}$ is subspace of R^n

column space of a matrix :-

The column space of an $m\times n$ matrix A written as col$A$ is set of all linear combinations of columns of A.

if $A = [a_1, a_2, \dots a_n]$ then

$$\text{col} A = \text{Span}\{a_1, a_2, \dots a_n\}.$$

col$A$ is subspace of R^m

→ find A if $w = \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix}$ is column space of A.

$$w = a\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = av_1 + bv_2.$$

$$A = [v_1 \ \ v_2]$$

$$w = \text{Span}\left\{\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}\right\}$$

$$A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$$

Row Echelon form :-

$$\begin{bmatrix} 1 & 5 & 6 \\ 0 & 8 & 4 \\ 0 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 2 & 4 & 5 \\ 0 & 0 & 0 \end{bmatrix}$$

↓

(i) The number zeroes before non zero element in a row is less than the number of such zeroes in next row.

(ii) if any zero rows ⇒ They must come after non zero rows.

(iii) leading element of non zero row comes after the zero elements of that row

Then matrix is in row echelon form.

General Example :-

$$\begin{bmatrix} 2 & a & b & c & d & e \\ 0 & 5 & f & g & h & i \\ 0 & 0 & 0 & j & k & l \\ 0 & 0 & 0 & 0 & m & n \\ 0 & 0 & 0 & 0 & 0 & p \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon form :-

REF with → leading entry of non zero row 1 and all entries abows and below of leading entries are zeros

if $A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 2 \\ 6 & 7 & 3 \end{bmatrix}$

↓

Reduced row echelon ⇒ $\begin{bmatrix} 1 & 0 & 10 \\ 0 & 1 & 8 \end{bmatrix}$

Steps to be followed to convert a matrix into RREF

you get Row echelon form

Step-1:- choose 1st non zero entry in 1st column and perform row operations to make all entries below leading entry zero

Step-2:- move to next ~~column~~ rows and repeat steps till all ~~columns~~ rows.

Step-3:- ~~use~~ ~~convert~~ change the leading entries to one by dividing the row with leading entry itself

Step-4:- using this leading entry 1 to make elements above the leading entry zero

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4 & 5 & 4 \\ 6 & 7 & 3 \end{bmatrix} \rightarrow 0 \ 1$$

zer 2, 1, 3

$R_2 \rightarrow R_2 - 2R_1$

$R_3 \rightarrow R_3 - 3R_1$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 4-4 & 5-6 & 4-2 \\ 6-6 & 7-21 & 3-3 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & -14 & 0 \end{bmatrix}$$

$R_3 \rightarrow R_3 - 14R_2$

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & -28 \end{bmatrix} \rightarrow \text{Row echelon form}$$

RREF:-

$R_1 \rightarrow R_1/2$

$R_2 \rightarrow R_2/-1$

$R_3 \rightarrow R_3/-28$

$$A = \begin{bmatrix} 1 & 3/2 & 1/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \rightarrow R_1 - 3/2\, R_2$

$$A = \begin{bmatrix} 1 & 0 & \frac{1}{2}+3 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 0 & 5/2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

reduced row echelon form.

$R_2 \rightarrow R_2 + 2R_3$ $\quad R_1 \rightarrow R_1 - \frac{5}{2}R_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(a) $A = \begin{bmatrix} 1 & -3 & -2 \\ -5 & 9 & 1 \end{bmatrix}$ $U = \begin{bmatrix} 5 \\ 3 \\ -2 \end{bmatrix}$

$$AU = \begin{bmatrix} 1\times5 + (-3)(3) + (-2)(-2) \\ (-5)(5) + 9(3) + 1(-2) \end{bmatrix}$$

$$= \begin{bmatrix} 5-9+4 \\ -25+27-2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$AU = 0$$

$\therefore$ U is nullspace of A Null A = U

④ find Spaning Set for null Space of matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

let Null Space of A is $X \Rightarrow AX = 0$

Perform $R_2 \to 3R_2 + R_1$ $R_3 \to 3R_3 + 2R_1$ for A

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 5 & 10 & -10 \\ 0 & 0 & 13 & 26 & -26 \end{bmatrix}$$

$R_2 \to R_2/5$ $R_3 \to \frac{R_3}{13}$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 1 & 2 & -2 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1/-3$

$$A = \begin{bmatrix} 1 & -2 & \frac{1}{3} & -\frac{1}{3} & 7/3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 1/3\, R_2$

$$A = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A is NO Reduced into Row echelon form

$$AX = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = 0$$

$x_1 - 2x_2 - x_4 + 3x_5 = 0$

$x_3 + 2x_4 - 2x_5 = 0$

$x_1 = 2x_2 + x_4 - 3x_5$

$x_3 = -2x_4 + 2x_5$

$$\therefore \bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$= x_2 V_1 + x_4 V_2 + x_5 V_3$$

$\therefore \bar{x}$ is combination of $\{v_1, v_2, v_3\}$

So $\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is Spaning i.e $\bar{X}$

Set for Null Space of A

(5) Define Basis and find Basis for Null A and Col A where

Basis definition
[illegible]
pg [illegible]
[illegible]

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{bmatrix}$$

$R_2 \to R_2 + R_1 \quad R_3 \to 2R_3 \mp 3R_1$

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 2+(-2) & -6+4 & -3+(-2) & 1+(-4) \\ -6 \mp (-6) & 16 \mp (12) & 4 \mp (-6) & -6 \mp (-12) \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & 4 & -2 & -4 \\ 0 & -2 & -5 & -4 \\ 0 & 4 & 10 & 6 \end{bmatrix}$$

$R_1 \to R_1/-2, \quad R_3 \to R_3 + 2R_2$

$$A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & -2 & -5 & -4 \\ 0+0 & 4-4 & 10-10 & 6-8 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & 1 & 2 \\ 0 & -2 & -5 & -4 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad R_1 \to R_1 - R_2$$

$$A = \begin{bmatrix} 1 & 0 & 6 & 8 \\ 0 & -2 & -5 & -4 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2/-2 \\ R_3 \to R_3/-2 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 6 & 8 \\ 0 & 1 & 5/2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 \to R_1 - 8R_3 \\ R_2 \to R_2 - 2R_3 \end{matrix}$$

$$A = \begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ Reduce

$$A = \begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Nullspace $A \Rightarrow AX = 0$

$$\begin{bmatrix} 1 & 0 & 6 & 0 \\ 0 & 1 & 5/2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \bar{0}$$

$$\begin{bmatrix} x_1 + 6x_3 + 0 \\ 0 + x_2 + 5/2 x_3 + 0 \\ x_4 \end{bmatrix} = 0$$

$x_1 + 6x_3 = 0$ $\quad x_4 = 0$

$x_2 + \frac{5}{2} x_3 = 0$

$x_1 = -6x_3$, $x_2 = -\frac{5}{2} x_3$ $\quad x_4 = 0$

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -6x_3 \\ -5/2 x_3 \\ x_3 \\ 0 \end{bmatrix} = 0$$

$$\bar{x} = x_3 \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{x} = \text{Span} \begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -12 \\ -5 \\ 2 \\ 0 \end{bmatrix}$$

$\therefore$ Basis $\begin{bmatrix} -6 \\ -5/2 \\ 1 \\ 0 \end{bmatrix}$ nullspace

~~Basis col A~~

Basis of Col A $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

(6) find Basis & Dimension of

$$H = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix} \right\}$$

$$H = s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$$

$H = sv_1 + tv_2$ where $v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ & $v_2 = \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix}$

$H = \text{Span}\{v_1, v_2\}$ So

$\{v_1, v_2\}$ Basis of H and

$\text{Dim } H = 2$

Find dimension of Subspace

$$H = \left\{ \begin{matrix} a-3b+6c \\ 5a+4d \\ b-2c-d \\ 5d \end{matrix} \right\}$$

$$H = a\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} + d\begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$$

$H = av_1 + bv_2 + cv_3 + dv_4$ where,

$v_1 = \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}$ $v_4 = \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix}$

$H = \text{Span}\{v_1, v_2, v_3, v_4\}$ but $v_3 = 2v_1$

So By Spaning Set theorem we may discard v_3 and the remaining v_1, v_3, v_4 vectors still spans H

i.e $H = \text{Span}\{v_1, v_3, v_4\}$

$\therefore$ $\dim H = 3$

(7) Define linearly Independent and linearly dependent find dimension of Subspace spanned by vectors

let $v_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} -3 \\ 6 \\ 0 \end{bmatrix}$ $v_3 = \begin{bmatrix} -2 \\ 3 \\ 5 \end{bmatrix}$ $v_4 \begin{bmatrix} -3 \\ 3 \\ 5 \end{bmatrix}$,

let H be sub space spanned by given vectors

$H = \text{span}\{v_1, v_2, v_3, v_4\}$

$\underline{\text{Basis}} \Rightarrow$ linear Independent

$\boxed{v_2 = -3v_1}$ $\Rightarrow$ v_2 · dependent

spanning $\underline{\text{theorem}}$ $H = \text{Span}\{v_1, v_3, v_4\}$

v_1, v_3, v_4. Independ

$\text{Basis} = \{v_1, v_3, v_4\} \Rightarrow$

Dimension Subspace $= 3$

definitions

8 The Set $\beta = \{1+t^2,\ t+t^2,\ 1+2t+t^2\}$.

8 is a basis for P_2 find co-ordinate vector of $P(t) = 1+4t+7t^2$ relative to β

co-ordinates :-

Suppose $\beta = \{b_1, b_2 \ldots b_n\}$ is basis of V and x is in V. The co-ordinates of x relative to basis β are Scalars $c_1, c_2 \ldots c_n$ such that

$X = c_1 b_1 + c_2 b_2 \ldots + c_n b_n$ respect to basis β

and $[X]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \rightarrow$ Vector R^n

standard basis $P_2 = \{1, t, t^2\}$

$$[P(t)]_\beta = ?$$

$P(t) = 1+4t+7t^2$ $\quad$ $\beta = \{1+t^2,\ t+t^2,\ 1+2t+t^2\}$ $\quad$ P_2

co-ordinate vector $\Rightarrow \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$, $v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ $v_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$P_2 = \text{Span}\{v_1, v_2, v_3\}$ $\quad$ $c_1, c_2, c_3 \in R$

$P(t) = c_1 v_1 + c_2 v_2 + c_3 v_3$

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} c_1 + 0 + c_3 \\ c_2 + 2c_3 \\ c_1 + c_2 + c_3 \end{bmatrix} \Rightarrow \begin{aligned} c_1 + c_3 &= 1 \\ c_2 + 2c_3 &= 4 \\ c_1 + c_2 + c_3 &= 7 \end{aligned}$$

$c_1 + c_3 = 1 \quad -①$

$c_2 + 2c_3 = 4 \quad -②$

$c_1 + c_2 + c_3 = 7 \quad -③$

Solving ① & ③ ③ - ①

$$\begin{array}{r} c_1 + c_2 + c_3 = 7 \\ -c_1 \qquad -c_3 = -1 \\ \hline \boxed{c_2 = 6} \end{array}$$

now sub c_2 in ②

$6 + 2c_3 = 4 \Rightarrow 2c_3 = 4 - 6 \Rightarrow 2c_3 = -2$

$\boxed{c_3 = -1}$

Sub c_3 in eqn ①.

$c_1 + c_3 = 1$.

$c_1 - 1 = 1 \Rightarrow c_1 = 1 + 1 = 2 \Rightarrow \boxed{c_1 = 2}$

$$[p(t)]_B = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$