5 Sem maths unit-2

(9) State and prove Rank theorem

(1) Find the base for the row space of the matrix $A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$

(2) If $A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$ then find rank A and dim Null A

(3) Let $\beta = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for V and suppose $b_1 = 6c_1 - 2c_2$, $b_2 = 9c_1 - 4c_2$. Then find (i) $\underset{C \leftarrow B}{P}$ (ii) find $[x]_c$ for $x = -3b_1 + 2b_2$

(4) Is $\lambda = 4$ eigen value of $\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$? if so find one corresponding eigen vector.

(5) show that eigen values of Triangular matrix are entries on its main diagonal

(6) find characterstic equation of

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ also find algebraic multiplicity of eigen values

(7) show that if $v_1, v_2 \ldots v_r$ are eigen vectors that correspond to eigen values $\lambda_1, \lambda_2, \ldots \lambda_6$ of an $n \times n$ matrix A then the set $\{v_1, v_2 \ldots v_n\}$ is linearly Independent.

(8) find characterstic polynomial of matrix

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \text{ or } A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$$ & eigen values

Row space of matrix :- A is $m \times n$ matrix set of linear combinations of A is Row A

$$A = \begin{bmatrix} 1 & 2 \\ 5 & 6 \end{bmatrix} \quad r_1 = (1, 2), \quad r_2 = (5, 6)$$

$$\text{Row } A = c_1 r_1 + c_2 r_2$$

$$\text{Row } A = \text{Span}\{r_1, r_2\}$$

Rank

eigen-values and Eigen vectors

Eigen values and Eigen vectors

change of Basis

characteristic Equation

(9) find eigen values and eigen vectors of

$$A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$$

Vector space :- Set of vectors defined on addition and multiplication by Scalars.

$V = \{v_1, v_2, \ldots v_n\}$ & $v_1 + v_2 \in V$

$cV_1 \in V$

Linear combinations :-

$V = c_1 v_1 + c_2 v_2 + c_3 v_3 + \ldots c_n v_n$

$V = \text{Span}\{v_1, v_2, v_3, v_4\}$

$v_1 = \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix}$ $v_2 = \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix}$ $A = \{v_1, v_2\}$

$A = \begin{bmatrix} 2 & 3 \\ 5 & 8 \\ 7 & 1 \end{bmatrix} = [v_1 \ v_2] \Rightarrow A_{3\times2}$

Col A :- linear combinations of columns of A

$\text{Col}A = \text{Span}\{v_1, v_2\} = c_1 v_1 + c_2 v_2$

$\text{Col}A = c_1 \begin{bmatrix} 2 \\ 5 \\ 7 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 8 \\ 1 \end{bmatrix} = \begin{bmatrix} 2c_1 + 3c_2 \\ 5c_1 + 8c_2 \\ 7c_1 + c_2 \end{bmatrix}_{3\times1}$

$\text{Col}A = R^3 \Rightarrow A_{m\times n} \Rightarrow \text{Col}A = R^m$

Dimension :- Number vectors in ~~Column spot~~ Basis of matrix / vectorspace

Basis :- $B = \{b_1, b_2 . b_n\}$ is a Set of vectors which are linearly Independent. and B becomes basis of vectorspace if it spans that vectorspace.

i.e $V = \text{Span}\{b_1, b_2 \ldots b_n\} = S_1 b_1 + S_2 b_2 + \ldots S_n b_n$

Null space :- $\boxed{Ax = 0}$ of a matrix A

then x is called its Null space of A

Null A $\rightarrow R^n$ matrix if $A_{m\times n}$ matrix

The Rank Theorem :- Rank Nullity Theorem

(1) State and prove Rank Theorem :-

Statement :- The dimension of column space and Row space of an $m \times n$ matrix A are equal, the rank A, is also equal to number of pivot positions in A and satisfies the following equation.

$$\boxed{\text{Rank } A + \dim \text{Nul } A = n}$$

→ number of Non pivot Positions gives dim Nul A

Proof :- The dimension of Col$A$ of matrix A gives Rank A.

The pivot columns of matrix A form basis for col A which is its dimension.

→ The Rank A is number of pivot columns in A. which comes from pivot positions of echelon form of matrix A.

→ further, the rows of pivot position in echelon form of A forms basis for row space A which is Rank A so Rank A is dimension of row space.

Thus dimension of Row A = dimension col A = Rank A.

→ dimension of Nul A ⇒ The number of variables in $\boxed{\text{col } AX = 0}$ or In other words.

dimension of Nul A is equal to number of non pivot columns.

→ Thus pivot & non pivot columns Actually came from columns of A so, $\boxed{\text{number of pivot columns} + \text{num of non piv col} = n}$

$$\boxed{\text{Rank } A + \dim \text{Nul } A = n}$$

① if 7×8 matrix has rank 3, find dim Nul A
dim Row A and Rank A^T

Rank of a matrix - The rank of matrix A is dimension of column space or dimension of its Row space.

$\dim \text{Col} A = \dim \text{Row} A = \text{Rank} A$

$\boxed{\text{Rank} A = \text{Rank} A^T}$ $\text{Rank} A = \text{Rank} A^T$

given :- 3×8 matrix rank = 3

dim Row A = 3

Rank A^T = 3

dim Null A = 5

$A = 3\times 8 \Rightarrow$ Rank Thm $\text{Rank} A + \dim \text{Nul} A = n$

$3 + \dim \text{Nul} A = 8$

$\dim \text{Nul} A = 5$

② find the basis for Row A and Col A and Null A for matrix A = Rank A

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

$R_2 \to 2R_2 + R_1$ $R_3 \to 2R_3 + 3R_1$ $R_4 \to 2R_4 + R_1$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 7 & -14 & 14 & -49 \\ 0 & 9 & -18 & 10 & -23 \end{bmatrix}$$

$2R_2$	R_3+3R_1	$2R_4+$
2-2	6-6	2-2
6-5	22-15	14-5
-10+8	-38+24	-26+8
2+0	14+0	10+0
10-17	2-51	-6-17

$R_3 \to R_3/7$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 9 & -18 & 10 & -23 \end{bmatrix}$$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 9 & -18 & 10 & -23 \end{bmatrix}$$

$R_3 \to R_3 - R_2$ $\quad R_4 \to R_4 - 9R_2$

$R_3 - R_2$: $0-0$, $1-1$, $-2+2$, $2-2$, $-7+7$

$R_4 - 9R_2$: $0-0$, $9-(+9)$, $-18-(-18)$, $10-(18)$, $-23-(-63)$, $-23+63$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -8 & \cancel{26}\ 40 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & -8 & 40 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$ Row n

$R_1 \to R_1 + 5R_2$

$$\begin{bmatrix} -2 & 0 & -2 & 10 & -52 \\ 0 & 1 & -2 & 2 & -7 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1/-2$ $\quad R_3 \to R_3/-8$

~~$A = [+1 \quad +5/2 \quad -4 \quad 0 \quad 17/2]$~~

$\to R_1 \to R_1 - 10R_3 \quad R_2 \to R_2 - 2R_3$

$$\begin{bmatrix} -2 & 0 & -2 & 0 & -2 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 - 10R_3$: $-52-(-50)$, $-52+50$

$-7-2(-5)$, $-7+10$

$R_1 \to -2 \Rightarrow$

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Reduced row echelon
1. leading 1
2. above pivot '0' below
3. No of zeroes in a column is more than it's previous column

$$AX = 0 \Rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}}_{4\times5} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{5\times1} = 0$$

$$\begin{bmatrix} x_1 + x_3 + 0 + x_5 \\ 0 + x_2 - 2x_3 + 3x_5 \\ x_4 - 5x_5 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + x_3 + x_5 &= 0 \\ x_2 - 2x_3 + 3x_5 &= 0 \\ x_4 - 5x_5 &= 0 \end{aligned}$$

$x_1 = -x_3 - x_5$
$x_2 = +2x_3 - 3x_5$
$x_4 = 5x_5$

$$\text{Null } A = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

$= x_3 v_1 + x_5 v_2$

Null A = Span$\{v_1, v_2\}$

⇓

Basis

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$ basis for Nul

Pivot columns $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ or forms basis for ColA

dimension ColA = 3

dimen NulA = 2 $\boxed{\text{Rank } A = 3}$ 4×5

RT :- dim ColA + dim NulA = 5

3 + 2 = 5

(1, 0, 1, 0 1) (0, 1, -2, 0 3) (0 0 0 1, -5)

form basis for Row A

$\begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix}$ .. forms basis for ColA

1, 3, 5, 1

Eigen values and Eigen vectors

A be a n×n matrix a Scalar λ is called eigen value of A, if there exists a nontrival solution x of $\boxed{Ax = \lambda x}$ such X is called eigen vector corresponding to λ.

Characterstic equation :- $\det(A - \lambda I) = 0$ is called characterstic equation.

→ $\det(A - \lambda I)$ or $|A - \lambda I|$ is a polynomial of degree n in λ & is called characterstic polynomial of A & λ is eigen value if λ is root of polynomial

(3) find rank A and dim Nul A.

(2)

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ -1 & 2 & -4 & 1 \\ 5 & -6 & 10 & 7 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 - 5R_1$

$$\begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 14 & -35 & 42 \end{bmatrix} \begin{matrix} -5R_1 \\ 20 \\ -45 \\ 35 \end{matrix}$$

$\Rightarrow R_3 \to R_3/7$

$$A = \begin{bmatrix} 1 & -4 & 9 & -7 \\ 0 & -2 & 5 & -6 \\ 0 & 2 & -5 & 6 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2 \quad R_3 \to R_3 + R_2$

$$A = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & -2 & 5 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 / -2$

$$A = \begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Rank = A = 2

dim col A = 2

Basis col A $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ Basis

$Ax = 0$

$$\begin{array}{cccc} c_1 & c_2 & & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & -1 & 5 \\ 0 & 1 & -5/2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}_{3\times 4} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}_{4\times 1} = 0$$

$x_1 - x_3 + 5x_4 = 0$

$x_2 - \frac{5}{2}x_3 + 3x_4 = 0$

$x_1 = x_3 - 5x_4$

$x_2 = \frac{5}{2}x_3 - 3x_4$

dim Nul A = 2

$$x_3 \underbrace{\begin{bmatrix} 1 \\ 5/2 \\ 1 \\ 0 \end{bmatrix}}_{v_1} + x_4 \underbrace{\begin{bmatrix} -5 \\ -3 \\ 0 \\ 1 \end{bmatrix}}_{v_2}$$

Basis

$$\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_3 - 5x_4 \\ \frac{5}{2}x_3 - 3x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

(4) Is $\lambda = 4$ an eigen value of

$\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}$ if so find corresponding eigen vector.

if $Ax = \lambda x \Rightarrow \lambda$ is eigen value

$(A - \lambda I)x = 0$ here $\lambda = 4 \Rightarrow (A - 4I)\bar{x} = 0$

$\rightarrow$ character eqn $|A - \lambda I| = 0 \Rightarrow \lambda$ is eigen value

$$\det\left(\begin{bmatrix} 3 & 0 & -1 \\ 2 & 3 & 1 \\ -3 & 4 & 5 \end{bmatrix}_{3\times 3} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 3-4 & 0 & -1 \\ 2 & 3-4 & 1 \\ -3 & 4 & 5-4 \end{vmatrix} = \begin{vmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{vmatrix}$$

$\Rightarrow -1[-1-4] + 0 - 1(8-3)$

$\Rightarrow 5 - 5 = 0$ λ^3

$\therefore$ The characteristic $|A - \lambda I| = 0$ at $\lambda = 4$

SO $\lambda = 4$ is eigen values

$$\begin{bmatrix} -1 & 0 & -1 \\ 2 & -1 & 1 \\ -3 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_2 \to R_2 + 2R_1 \quad R_3 \to R_3 - 3R_1$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$R_3 \to R_3 + 4R_2$

$$\begin{bmatrix} -1 & 0 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} -x_1 - x_3 \\ -x_2 - x_3 \end{bmatrix} = 0 \Rightarrow -x_1 - x_3 = 0 \quad x_1 = -x_3$$

$$-x_2 - x_3 = 0 \Rightarrow x_2 = -x_3$$

$\Rightarrow$ eign $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is eign vector corresponding eign vale $\lambda = 4$

(5) show that eigen values of a Triangular matrix are entries of its diagonals

Statement :- The eigen values of a Triangular matrix are entries on its main diagonal

proof :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}_{3\times3}$ be an upper Triangular matrix

→ Let λ be a Scalar.

→ The Scalar λ is eigen value of A if and only if $(A-\lambda I)\bar{x} = 0$ has non Trival Solution,

i.e $\bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \neq 0$

$$(A-\lambda I)\bar{x} = 0$$

$$\left[\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right]\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} \\ 0 & a_{22}-\lambda & a_{23} \\ 0 & 0 & a_{33}-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$(a_{11}-\lambda)x + a_{12}x_2 + (a_{13})x_3 = 0 \quad -(1)$$

$$0 + (a_{22}-\lambda)x_2 + a_{23}x_3 = 0 \quad -(2)$$

$$(a_{33}-\lambda)x_3 = 0 \quad -(3)$$

$(a_{11}-\lambda)x_1 + a_{12}x_2 + a_{13}x_3 = 0$ — (1)

$(a_{22}-\lambda)x_2 + a_{23}x_3 = 0$ — (2)

$(a_{33}-\lambda)x_3 = 0$ — (3)

$x_1 \neq 0$. $x_2 \neq 0$ $x_3 \neq 0$ $\qquad \begin{bmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{bmatrix}$

if $\lambda \neq a_{11}$ $\quad \lambda \neq a_{22}$ $\quad \lambda \neq a_{33}$

$(a_{33}-\lambda)x_3 = 0$ $\qquad$ [ab=0 a=0 b=0]

$a_{33}-\lambda = 0$. $\quad x_3 = 0$

$x_3 \neq 0 \Rightarrow a_{33}-\lambda = 0$, $\boxed{a_{33}=\lambda}$

if $x_3=0 \Rightarrow (A-\lambda I)\bar{x} = 0 \Rightarrow$ Trival sol^n

but $(A-\lambda I)\bar{x} = 0$ has non Trival sol^n

so. That possible only if $\lambda = a_{33}$

$\therefore \lambda = a_{11}, a_{22}, a_{33}$

$\therefore$ Hence we conclude that for a Triangular matrix eigen value λ is equal to diagonal entry of matrix A

7) find characterstic polynomial → దీని Zero కి equate చేయండి

6) Find characterstic Equation of → దీని Zero చేయడం i.e just $|A-\lambda I|$

6) 8)

$$A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and find multiplicities of eigen values

given $A = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

characterstic quation is $|A-\lambda I| = 0$.

$$A-\lambda I = \begin{bmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$|A-\lambda I| = \left|\begin{bmatrix} 5-\lambda & -2 & 6 & -1 \\ 0 & 3-\lambda & -8 & 0 \\ 0 & 0 & 5-\lambda & 4 \\ 0 & 0 & 0 & 1-\lambda \end{bmatrix}\right| = 0 \quad 4 \quad 2\times 2$$

$[(5-\lambda)(3-\lambda)-0]\,[0-0]\,[0-8]\,[(5-\lambda)(1-\lambda)-0] = 0$

$(5-\lambda)(3-\lambda)(5-\lambda)(1-\lambda) = 0 \; -①$ $\quad -8 \neq 0$

$5-\lambda = 0 \Rightarrow \lambda = 5 \quad 3-\lambda = 0 \quad \lambda = 3 \quad \lambda = 5 \quad 1 = \lambda$

$$\boxed{\lambda = 5, 5, 3, 1}$$

$(5-\lambda)^2(3-\lambda)(1-\lambda) = 0$

$(5^2+\lambda^2-10\lambda)(3-3\lambda-\lambda+\lambda^2) = 0$

$(\lambda^2+25-10\lambda)(\lambda^2-4\lambda+3) = 0$

$\lambda^4 - 4\lambda^3 + 3\lambda^2 + 25\lambda^2 - 100\lambda + 75 - 10\lambda^3 + 40\lambda^2 - 30\lambda =$

$$\boxed{\lambda^4 - 14\lambda^3 + 68\lambda^2 - 130\lambda + 75 = 0}$$

$A \to 4\times 4 \quad \lambda^4$

$\lambda = 5$ multip $= 2$

$\lambda = 3$ multip 1

$\lambda = 1$ multi 1

Theorem :- show that eigen vector are linearly Independent

6 Statement :- If $v_1, v_2, \dots v_n$ are eigen

7 vectors that correspond to eigen values $\lambda_1, \lambda_2 \dots \lambda_n$ of an $n \times n$ matrix A then set $\{v_1, v_2 \dots v_n\}$ is linearly Independent

Linearly Independent :-

proof :- given, A is $n \times n$ matrix let $\lambda_1, \lambda_2 \dots \lambda_n$ are distinct eigen values of A That correspond to eigen vectors $v_1, v_2, \dots v_n$.

→ Let us Assume that, $\{v_1, v_2 \dots v_n\}$ is linearly dependent

By linear dependency theorm then any vector v_{p+1} can be written as linear combination of its preceeding vectors $v_1, \dots v_p$

$$v_{p+1} = c_1 v_1 + c_2 v_2 + \dots c_p v_p \quad \text{---} (1)$$

$c_1, c_2, \dots c_p$ not all zeores

multiplying (1) with A on B.S

$$A v_{p+1} = c_1 A v_1 + c_2 A v_2 + \dots c_p A v_p$$

from, $\boxed{A\bar{x} = \lambda \bar{x}}$ $A v_i = \lambda_i v_i$ [for $i = 1, 2 \dots n$]

$$\lambda_{p+1} v_{p+1} = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 \dots c_p \lambda_p v_p \quad (2)$$

Again multip eq (1) with λ_{p+1} on B.S

$$\lambda_{p+1} v_{p+1} = c_1 \lambda_{p+1} v_1 + c_2 \lambda_{p+1} v_2 + \dots c_p \lambda_{p+1} v_p \quad (3)$$

eq (2) − eq (3)

$0 = c_1 v_1 (\lambda_1 - \lambda_{p+1}) + c_2 v_2 (\lambda_2 - \lambda_{p+1}) \cdots + c_p v_p (\lambda_p - \lambda_{p+1})$

$0 = c_1 (\lambda_1 - \lambda_{p+1}) v_1 + c_2 (\lambda_2 - \lambda_{p+1}) v_2 \cdots + c_p (\lambda_p - \lambda_{p+1}) v_p$

Since $v_1 . v_2 \ldots v_p$ are Independend vectors

then $c_1(\lambda_1 - \lambda_{p+1}) = 0$ $\quad c_2(\lambda_2 - \lambda_{p+1}) = 0$

$c_p(\lambda_p - \lambda_{p+1}) = 0$.

λ_i distinct values $c_1 = 0$ $\quad c_2 = 0$ $\quad c_p = 0$

but this is contradiction to ~~the Assum~~ p

$c_1, c_2 \ldots c_p$ not all zeroes

but then $\{v_1 \ldots v_n\}$ Linearly dependt

is contradict

contradiction rised due to assumption

$v_1, v_2 \ldots v_n$ linerly dependent

$\therefore$ our assumption was wrong

$v_1, v_2 \ldots v_n$ are Linearly Independent

vectors

(8) find characteristic polynomial of $\lambda = 4$

(9) $A = \begin{bmatrix} -4 & -1 \\ 6 & 1 \end{bmatrix}$ and find its eigen values eigen vector

Characteristic polynomial is $|A - \lambda I| = 0$

$$A - \lambda I = \begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix}$$

$$|A - \lambda I| = [(-4-\lambda)(1-\lambda) - 6(-1)] = 0$$

$$-4 + 4\lambda - \lambda + \lambda^2 + 6 = 0$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$\lambda^2 + 2\lambda + \lambda + 2 = 0$$

$$\lambda(\lambda+2) + 1(\lambda+2) = 0$$

$$\lambda + 1 = 0 \qquad \lambda + 2 = 0$$

$$\boxed{\lambda = -1} \qquad \boxed{\lambda = -2}$$

eigen values are $\lambda = -1, -2$.

Corresponding eigen vectors :-

$$(A - \lambda I)\bar{X} = 0$$

$$\begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad -(1)$$

at $\lambda = -1$ $\begin{bmatrix} -4+1 & -1 \\ 6 & 1-(-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$$\begin{bmatrix} -3 & -1 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_2 \to R_2 + 2R_1$

$$\begin{bmatrix} -3 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow -3x_1 - x_2 = 0$$

$$x_2 = -3x_1$$

$$\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -3x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ eigen vector at } \lambda = -1$$

at $\lambda = -2$ $\quad (A - \lambda I)\bar{X} = 0$

$$\begin{bmatrix} -4-\lambda & -1 \\ 6 & 1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -4+2 & -1 \\ 6 & 1-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\begin{bmatrix} -2 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$R_2 \to R_2 + 3R_1$

$$\begin{bmatrix} -2 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$-2x_1 - x_2 = 0$$

$$x_2 = -2x_1$$

eign vector $\bar{X} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ -2x_1 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\therefore$ at $\lambda = -2$ eigne vecto $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$

$\lambda = -1$ eigen vec $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$A \Rightarrow 2 \times 2$

(3) Let $B = \{b_1, b_2\}$ and $C = \{c_1, c_2\}$ be bases for vector space V and suppose $b_1 = 6c_1 - 2c_2$ and $b_2 = 9c_1 - 4c_2$

a) find change of cordinate matrix from B to C

b) find $[\bar{x}]_C$ for $x = -3b_1 + 2b_2$.

$b_1 = 6c_1 - 2c_2 \quad b_2 = 9c_1 - 4c_2$

$b_1 = \text{span}\{c_1, c_2\} \quad b_2 = \text{span}\{c_1, c_2\}$

c_1 and c_2 vector basis for V

$[b_1]_C = \begin{bmatrix} 6 \\ -2 \end{bmatrix} \quad [b_2]_C = \begin{bmatrix} 9 \\ -4 \end{bmatrix}$

Change of co-ordinate matrix B to C.

$$\underset{C \leftarrow B}{P} = \begin{bmatrix} [b_1]_C & [b_2]_C \end{bmatrix} = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix}$$

b) $[\bar{x}_C]$ for $x = -3b_1 + 2b_2$.

$$[x]_B = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$[\bar{x}]_C = [-3b_1 + 2b_2]_C$$

$$[\bar{x}]_C = -3[b_1]_C + 2[b_2]_C$$

$$\Rightarrow [\bar{x}]_C = \begin{pmatrix} -3 \\ 2 \end{pmatrix} \begin{bmatrix} [b_1]_C & [b_2]_C \end{bmatrix}$$

$$= \underset{C \leftarrow B}{P} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$[\bar{x}]_C = \begin{bmatrix} 6 & 9 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} -3 \\ 2 \end{bmatrix}$$

$$[\bar{x}]_C = \begin{bmatrix} -18 + 18 \\ +6 - 8 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

$$[\bar{x}]_C = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Basis :- $B = \{b_1, b_2 \ldots b_n\}$ is a basis for V if $b_1, b_2 \ldots b_n$ are linearly independent and $V = \text{span}\{b_1, b_2, \ldots b_n\}$

co-ordinates :- if $\bar{x}$ is a vector in V. and ~~whose~~ basis of V is $B = \{b_1, b_2, \ldots b_n\}$

then ~~$x = c_1 b_1 + c_2 b_2$~~

$$x = \text{span}\{b_1, b_2, \ldots b_n\}$$

$$x = c_1 b_1 + c_2 b_2 + \ldots c_n b_n$$

$\Rightarrow$ $[x]_B = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ $\Rightarrow$ The $c_1, \ldots c_n$ are co-ordinates of $\bar{X}$ at Basis B

(co-ordinate vector)

$\Rightarrow$ A Vector space can have more than one Basis. The co-ordinate vectors when a vector Transformed from one basis to other is change of Bases.