3rd chapter Differentiation

Date ____
Page ____

(1) LAQ's

(1) f and g are differentiable functions at a and c be any Scalar
=) ST cf, f+g, fg and f/g are differentiable at a exept f/g if g(a) = 0 and

(i) $(cf)'(a) = cf'(a)$

(ii) $(f+g)'a = f'a + g'(a)$

(iii) $(fg)'a = f(a)g'(a) + f'(a)g(a)$

(iv) $(f/g)'(a) = \dfrac{f'(a)g(a) - f(a)g'(a)}{g^2(a)}$

if $g(a) \neq 0$

(2) State and prove Rolles theorem

(3) State & prove MVT or Lagrangens Mean value Theorem

(4) If f is differentiable at a and g is differentiable at f(a) =) ST gof differentiable at a and $gof'(a) = g'f(a) \cdot f'(a)$

(5) If f is defined on an open intervel containing x_0,

2

f assumes its maximum / minimum at x_0 and f is differentiable at x_0

$\Rightarrow$ ST $f'(x_0) = 0$.

6. State and prove generalized (or) cachy's mean value Theorem

7. f is differential at $a \Rightarrow$ ST f is continus at a

8. find Tylor Series for $\cos x$.
   define Tylor Series

9. PT $\lim_{x \to 0} \frac{\cos x - 1}{x^2} = \frac{-1}{2}$.

10. find the limits if they exists.
   $\lim_{x \to 0} \frac{e^{2x} - \cos x}{x}$, $\lim_{x \to \infty} \frac{x^3}{e^{2x}}$, $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

11. find Taylor Series for $\sinh x = \frac{1}{2} e^{x} - e^{-x}$

12. let $f(x) = x^2$ for $x \geq 0$ $f(x) = 0$ for $x < 0$
   $\Rightarrow$ (i) check graph of
   (ii) Show that f is differentiable at $x = 0$

13. ST $\sin x \leq x$ for all $x \geq 0$

3

① Derivative of a function 'f' at a point 'a'

If f is a real valued function defined on an open interval containing a point 'a' then f is said to have derivative at 'a' if the limit

$$\lim_{x \to a} \frac{f(x)-f(a)}{x-a} \text{ exists \& is finite.}$$

$\Rightarrow$

$$f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

where

$f'(a)$ = derivative of f at a.

Date ______ Page ______

(1) LA-① If f and g are two differentiable functions at point 'a' then PT the following functions are also differentiable at point a.

(i) $(cf)' a = cf'(a)$.

Take $(cf)'(a) = \lim_{x \to a} \frac{c \cdot f(x) - (cf)(a)}{x - a}$
$= \cancel{c \cdot \lim_{x \to a}}$

$= \lim_{x \to a} c \cdot \frac{f(x) - f(a)}{x - a}$

$= c \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$

$(cf)'(a) = c \cdot f'(a)$

(ii) $(f+g)'(a) = f'(a) + g'(a)$

$(f+g)' a = \lim_{x \to a} \frac{(f+g)(x) - (f+g)(a)}{x - a}$

$= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} \frac{g(x) - g(a)}{x - a}$

$= f'(a) + g'(a)$

$\therefore (f+g)' a = f'(a) + g'(a)$

Date
Page

5

(iii) $(fg)'(a) = \lim_{x \to a} \frac{(fg)(x) - (fg)(a)}{x - a}$

$= \lim_{x \to a} \frac{(fg)(x) - f(x)g(a) + f(x)g(a) - (fg)(a)}{x - a}$

$= \lim_{x \to a} f(x) \frac{g(x) - g(a)}{x - a} + \lim_{x \to a} g(a) \frac{f(x) - f(a)}{x - a}$

$= f(a)\, g'(a) + g(a)\, f'(a)$
$\forall x \in \text{dom } fg \quad x \neq a$

(iv) $\left(\frac{f}{g}\right)' a = \frac{[g(a) f'(a) - f(a) g'(a)]}{g^2(a)} \quad g(a) \neq 0$

$g(a) \neq$ & g is continution at a
$\Rightarrow$ there exists a open intervel
I containing a such that $g(x) \neq 0$
for $x \in I$

$\left(\frac{f}{g}\right) x - \left(\frac{f}{g}\right) a = \frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}$

$= \frac{g(a) f(x) - f(a) g(x)}{g(x) g(a)}$

Date ______ Page ______

$$\left(\frac{f}{g}\right)x - \left(\frac{f}{g}\right)a = \frac{g(a)f(x) - g(a)f(a) + g(a)f(a) - f(a)g(x)}{g(x)\,g(a)}$$

$$\lim_{x\to a} \frac{\left(\frac{f}{g}\right)x - \left(\frac{f}{g}\right)a}{x-a} = \frac{1}{g(x)g(a)}$$

$$= \lim_{x\to a} \frac{\left[g(a)\frac{f(x)-f(a)}{x-a} - f(a)\frac{g(x)-g(a)}{x-a}\right]}{g(x)\,g(a)}$$

$$\lim_{x\to a} \left(\frac{f}{g}\right)x - \left(\frac{f}{g}\right)a = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$$

$$\therefore \left(\frac{f}{g}\right)' a = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}, \quad g(a) \neq 0$$

(8) ~~find~~ ?
(2) State and prove Rolle's Theorem

(2) 4m Statement:- If $f(x)$ is a function defined on [a b] and
$f(x)$ is continuos in [a b]
$f(x)$ is differentiable on (a b)
and if $f(a) = f(b)$, then there exists atleast one point $c \in (a, b)$ such that $f'(c) = 0$

proof:-
given that
$f(x)$ is continuos in [a, b]
and we know,
every continuos function is bounded
So $f(x)$ is bounded
$\Rightarrow$ $f(x)$ have maximum and minimum
we have 2 cases
(i) max & min values equal
(ii) max & min values different

fermat theorem

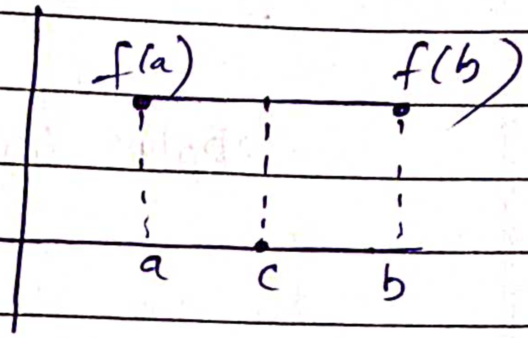
Case-(i) :- If maximum and minum values are equal then $f(x)$ is $f(a) = f(b)$

constant function

$f(x) = K$

$f'(x) = 0$

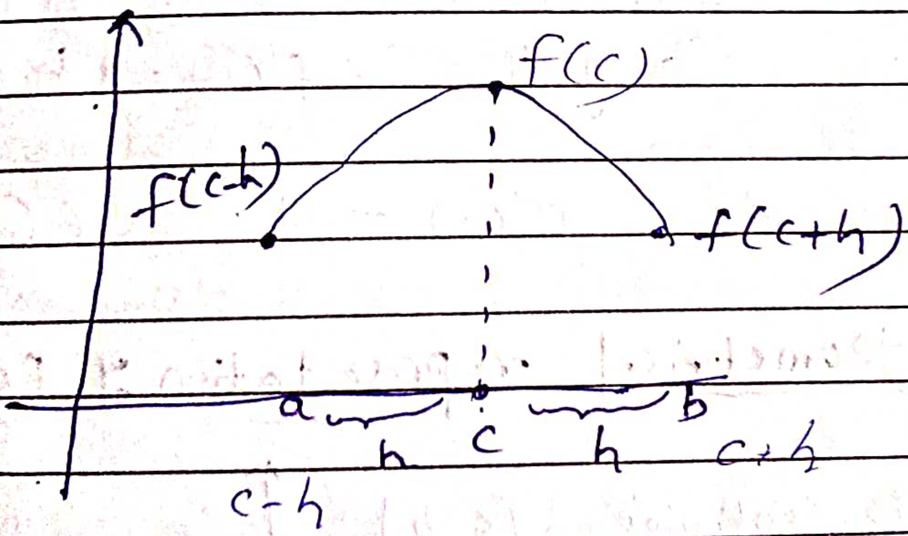
$\forall\, x \in [a\ b]$



Let $c \in (a\,b) \Rightarrow f'(c) = 0$

Case-(ii) :-

if max and minimum values are different suppose $f(x)$ is max at c.



Here $f(c+h) - f(c)$ and $f(c-h) - f(c)$ both are $-ve$ for a small value of h.

as f is differentiable.

$\frac{f(c+h)-f(c)}{h} \le 0$ and $\frac{f(c-h)-f(c)}{-h} \ge 0$

apply limits both sides $h \to 0$

$\lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \le 0$ & $\lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} \ge 0$

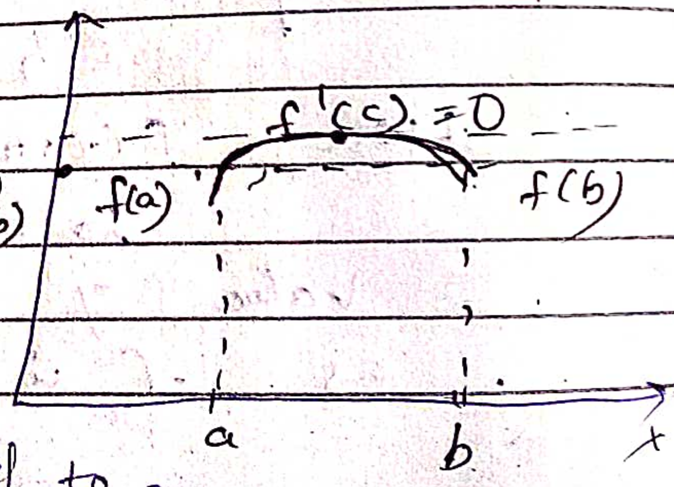
$R f'(c) \le 0$ $\quad L f'(c) \ge 0$

f is differentiable only if $R f'(c) = 0$ = $L f'(c) = 0$

$\therefore$ $f'(c) = 0$. $c \in (a,b)$

geometrical representation of Rolles Theorem

if f is continuos on [a b]
differentiable on (a b)
$f(a) = f(b) \Rightarrow \exists$ at least $f(a)$ $f(b)$
One $x \in (a\ b)$
$f'(c) = 0$.



at c Tangent at c parellel to x.

Date ____ Page ____

(7) State and prove lagranges mean Value Theorem

Statement :- $f(x)$ is function of x
and
(i) $f(x)$ is continuos on $[a\ b]$
(ii) $f(x)$ is differentiable on (a,b)
Then $\exists$ $c \in (a,b)$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

proof :- given
~~we know~~ $f(x)$ is continuos on $[a\ b]$
every polynomial is continuos & differentiable
let us Take a polynomial ~~x~~ ~~which is~~
continuos on $[ab]$ $\phi(x)$
$\Rightarrow$
let $\phi(x) = f(x) + Kx$ — (1)
we know $f(x)$, Kx continuos & differentiable
$\Rightarrow$ $\phi(x)$ also becomes
Continuos on $[a\ b]$ and
differentiable on (a,b)

Then we know according to roll's theorem if a function is differentiable on an open interval I and $f(a)=f(b)$ continous on closed Interval I then there exist a $c \in (a,b)$ where $\cancel{f'(c) = f(b)-f(a)}$

$$f'(c)=0$$

so

for function $\phi(x)$ on $[a\ b]$ since it is continuos & differentiable so $\exists$ a $c \in [a\ b)$ $\phi(a)=\phi(b)$

where $\phi'(c)=0$ $\left[\phi(x) = f(x)-Kx\right.$ $\phi'(x) = f'(x)-K$

$$f(c) + K0 = 0$$

$$f'(c) = K$$

$$\boxed{\Rightarrow K = f'(c)}$$

we have

$$K = \frac{f(b)-f(a)}{b-a}$$

$$\boxed{\text{So } f'(c) = \frac{f(b)-f(a)}{b-a}}$$

$\phi(x) = f(x) + Kx$

let if $\phi(a) = \phi(b)$

then

$f(a) + Ka = f(b) + Kb$

$Ka - Kb = f(b) - f(a)$

$-K(b-a) = f(b) - f(a)$

$$\boxed{-K = \frac{f(b) - f(a)}{b - a}}$$

Date ____
Page ____

(i) if f is differentiable at a and g is differentiable at $f(a) \Rightarrow$ ST gof differentiable at a

(ii) $(g \circ f)'(a) = g'[f(a)]\, f'(a)$

given f & g differentiable

$\Rightarrow f(x) \Rightarrow$ different at $a = 0 \Rightarrow$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \text{ exists}$$

$$g'(a) = \lim_{h \to 0} \frac{g(a+h) - g(a)}{h \to 0}$$

$\Rightarrow$ Composite function $\cancel{gof(a) = f}$

$g \circ f(x) = g(f(x))$

$$\Rightarrow g \circ f'(a) = \lim_{h \to 0} \frac{g \circ f(a+h) - g \circ f(a)}{h}$$

$$= \lim_{h \to 0} \frac{g(f(a+h)) - g(f(a))}{h} \times \frac{f(a+h) - f(a)}{f(a+h) - f(a)}$$

=

14

$$\lim_{h\to 0} \frac{g(f(a+h))-g(f(a))}{f(a+h)-f(a)} \cdot \frac{f(a+h)-f(a)}{h}$$

$$\lim_{h\to 0} \frac{g(f(a+h))-g(f(a))}{f(a+h)-f(a)} \lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

$gof'(a) = g'f(a) \cdot f'(a)$ $\qquad$ $gof'(a) = g'(f(a))f'(a)$

$gof'(x) = g'f(x) f'(x)$

(i) use chain rule & find $h = gof$

if $h(x) = (x^4+13x)^7$ find h'

$g(x) = x^7 \qquad f(x) = x^4+13x$

$gof = gof(x) = g(f(x)) = g(x^4+13x)$

$= (x^4+13x)^7$

$h' = gof'(x) = g'(f(x)) \cdot f'(x)$

$g(x) = x^7 \Rightarrow g'(x) = 7x^6 \Rightarrow g'(f(x)) = 7(x^4+13x)$

$f(x) = x^4+13x \Rightarrow f'(x) = 4x^3+13$

$h' = g'f(x) \cdot f'(x)$

$h' = 7(x^4+13x)^6 \cdot 4x^3+13$

Date
Page

$x^n = n x^{n-1}$

$h(x) = (x^4+13x)^7$ $\frac{d}{dx} f(x)^n = n f(x)^{n-1} \frac{d}{dx} f(x)$

$h'(x) = 7 \cdot (x^4+13x)^{7-1} \cdot \frac{d}{dx}(x^4+13x)$

$= 7 (x^4+13x)^6 [4x^3+13]$

(5) If f is defined on an open interval I containing x_0 if f assumes its maximum ~~and~~/ minimum at x_0 and f is differentiable at x_0

$\Rightarrow$ ST $f'(x_0) = 0$.

(5)

$\Rightarrow$ Let f an interval $(a\ b)$ & given $x_0 \in (a,b)$ and f is diferentiable at (a,b)

let us consider maximum at x_0 of function f

$\Rightarrow f'(x_0)$ — $f'(x_0) > 0$ or maximum ; $f'(x_0) < 0$ ✓ ; $f'(x_0) = 0$

Case-(i) $f'(x_0) > 0$

Since f is diferentiable at x_0

$$\Rightarrow f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

exists & finite.

$\Rightarrow f$ is differentiable $\Rightarrow$ then
f is continuous at x_0 $\epsilon - \delta$
$\Rightarrow$ There exists $\delta > 0$ and domain f
$\Rightarrow$ $|x - x_0| < \delta$ and $|f(x) - f(x_0)| < \epsilon$

$\delta > 0$, $a < x_0 < b$
$a < x_0 - \delta < x_0 + \delta < b$ & $0 < |x - x_0|$

$$\Rightarrow \frac{f(x) - f(x_0)}{x - x_0} > 0 \quad [f'(x) > 0]$$

$x_0 < x < x_0 + \delta$
$x_0 < x$
$f(x_0) < f(x)$

$\therefore f(x) > f(x_0)$

but $f'(x_0) > 0 \Rightarrow f$ maximum at x_0

$\therefore$ contrary to assumption f is maximum at x_0
$f'(x_0) \neq 0$

Case-(ii) $f'(x_0) < 0$

$\Rightarrow \quad f'(x_0) < 0, \quad \delta > 0$

$0 < |x - x_0| < \delta$

$$\frac{f(x) - f(x_0)}{x - x_0} < 0$$

$x_0 - \delta < x < x_0$

$f(x) > f(x_0) \quad f(x_0) < f(x_0) \quad f(x) < f(x_0)$

$f(x) \geq f(x_0) \quad f(x_0) \geq f(x)$

$\therefore$ contradiction.

$\therefore f'(x_0) \not< 0$

$\therefore f'(x) = 0$

Date ________
Page ________

$$\phi(a) = \phi(b)$$

$$f(a) + k\,g(a) = f(b) + k\,g(b)$$

$$f(a) - f(b) = \cancel{k}(g(b) - k g(a))$$

$$-(f(b) - f(a)) = k[g(b) - g(a)]$$

$$\Rightarrow -k = +\frac{(f(b) - f(a))}{g(b) - g(a)}$$

Now ~~if~~ $\phi(x)$ is continuos on $[a\ b]$ differentiable on (a, b) and $\phi(a) = \phi(b)$ so according to rolles Theorem we can ~~write~~. Then $c \in (a\ b)$ where

$$\cancel{\phi(x)} = \quad \phi'(c) = 0$$

$$\phi(x) = f(x) + k g(x) \Rightarrow \phi'(x) = f'(x) + k g'(x)$$

$$\phi'(c) = 0$$

$$\phi'(c) = f'(c) + k g'(c) = 0$$

$$-f'(c) = k g'(c)$$

$$k = \frac{-f'(c)}{g'(c)}$$

$$\cancel{-}\frac{[f(b) - f(a)]}{g(b) - g(a)} = \frac{\cancel{-}f'(c)}{g'(c)} \Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

(6) If f is differentiable at a
$\Rightarrow$ PT f is continuos at a

(7)

f differentiable a

$$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \text{ exists finite}$$

$$f(x) = f(x) - f(a) + f(a)$$

$$f(x) = (x-a)\frac{f(x)-f(a)}{x-a} + f(a)$$

$$\lim_{x\to a} f(x) = \lim_{x\to a} (x-a)\frac{f(x)-f(a)}{x-a} + \lim_{x\to a} f(a)$$

$$\Rightarrow \left[\lim_{x\to a} p(x)g(x) = \lim_{x\to a} p(x) \lim_{x\to a} g(x)\right]$$

finite

$$\lim_{x\to a} f(x) = \lim_{x\to a} (x-a) \cdot \lim_{x\to a}\frac{f(x)-f(a)}{x-a} + f(a)$$

$$\lim_{x\to a} f(x) = 0 + f(a) \qquad \left[\lim_{x\to a}(x-a) = 0\right]$$

$$\lim_{x\to a} f(x) = f(a)$$

$\therefore$ f is continuos at a

L Hospital rule

used for evaluating limits if :-

(i) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{0}{0}$ or $\frac{\infty}{\infty}$ } Indeterminate form

(ii) $f(x), g(x)$ differentiable & $g'(x) \neq 0$

$$\Rightarrow \lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

(Q) PT $\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \frac{-1}{2}$

(A) $\frac{0}{0}$ form so using L-Hospital rule.

$$\Rightarrow \lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx} x^2}$$

$$= \lim_{x\to 0} \frac{-\sin x}{2x}$$

use L-Hospital again

$$= \lim_{x\to 0} \frac{-\cos x}{2}$$

$$= \frac{-\cos 0}{2} = \frac{-1}{2}$$

7)

properly write it in This way for each problem

(a) $\lim_{x\to 0} \frac{x^3}{\sin x - x}$

10

Here $f(x) = x^3$

$g(x) = \sin x - x$.

since $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$

it gives $\frac{0}{0}$ Indeterminate form

so using L'Hospital Rule,

$$\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{f'(x)}{g'(x)}$$

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$g(x) = \sin x - x \Rightarrow g'(x) = \cos x - 1$

$$\Rightarrow \lim_{x\to 0} \frac{f'(x)}{g'(x)} = \lim_{x\to 0} \frac{3x^2}{\cos x - 1} = \frac{3(0)}{\cos 0 - 1}$$

$$= \frac{0}{0}$$

It is indeterminate form

so using

Date
Page

L-Hospital Rule again for,

$$\Rightarrow \lim_{x\to 0} \frac{3x^2}{\cos x-1}$$

$$\Rightarrow \lim_{x\to 0} \frac{6x}{-\sin x} \Rightarrow \frac{0}{0}$$ again Indeterminate form

Using L-Hospital Rule again.

$$\Rightarrow \lim_{x\to 0} \frac{6x}{-\sin x} \Rightarrow \lim_{x\to 0} \frac{6}{-\cos x} = \frac{6}{-\cos 0} = \frac{6}{-1}$$

$$\therefore \lim_{x\to 0} \frac{x^3}{\sin x - x} = -6.$$

(a) $\lim_{x \to 0} \frac{e^{2x} - \cos x}{x}$

here $f(x) = e^{2x} - \cos x$ $g(x) = x$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0$

using L-Hospital Rule

Consider $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{0}{0} \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim f'(x)}{g'(x)}$

$\lim_{x \to 0} \frac{e^{2x} - \cos x}{x} = \lim_{x \to 0} \frac{2e^{2x} + \sin x}{1}$

$= 2e^{2(0)} + \sin 0$

$= 2[e^0] + 0$

$= 2(1) + 0$

$\therefore \lim_{x \to 0} \frac{e^{2x} - \cos x}{x} = 2$

(c) $\lim\limits_{x \to \alpha} \dfrac{x^3}{e^{2x}}$

$f(x) = x^3 \quad g(x) = e^{2x}$

$\lim\limits_{x \to \alpha} \dfrac{x^3}{e^{2x}} \Rightarrow \lim\limits_{x \to \alpha} x^3 = \lim\limits_{x \to \alpha} e^{2x} = \alpha$

$\dfrac{\alpha}{\alpha}$ Indeterminate form

L-Hospital Rule.

$\Rightarrow \lim\limits_{x \to \alpha} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to \alpha} \dfrac{f'(x)}{g'(x)}$ $\boxed{e^{\alpha} = \alpha}$

$= \lim\limits_{x \to \alpha} \dfrac{3x^2}{2e^{2x}} = \dfrac{\alpha}{\alpha}$

Apply L-Hospital Rule..

$= \lim\limits_{x \to \alpha} \dfrac{3 \cdot 2x}{2 \cdot 2 \cdot e^{2x}} = \dfrac{\alpha}{\alpha}$

Apply L-Hospital Rule.

$\Rightarrow \lim\limits_{x \to \alpha} \dfrac{6(1)}{4 \cdot 2 \cdot e^{2x}} = \dfrac{6}{8e^{2(\alpha)}} = \dfrac{6}{\alpha} = 0$

$\lim\limits_{x \to \alpha} \dfrac{x^3}{e^{2x}} = 0$

(d) $\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x}$

$\boxed{\frac{d}{dx} \sqrt{x}^{1/2} = \frac{1}{2\sqrt{x}}}$

$\lim_{x \to 0} f(x) = \lim_{x \to 0} g(x) = 0 \longrightarrow \frac{0}{0}$

L-Hospital Rule :

$\Rightarrow \lim_{x \to 0} \frac{f(x)}{g(x)} = \lim \frac{f'(x)}{g'(x)}$

$\lim_{x \to 0} = \frac{\frac{d}{dx}\sqrt{1+x} - \frac{d}{dx}\sqrt{1-x}}{\frac{d}{dx} x}$

$\lim_{x \to 0} \frac{\frac{1}{2\sqrt{1+x}}[0+1] - \frac{1}{2\sqrt{1-x}}[0-1]}{1}$

$\lim_{x \to 0} \frac{1}{2\sqrt{1+x}} + \frac{1}{2\sqrt{1-x}} =$

$= \frac{1}{2\sqrt{1+0}} + \frac{1}{2\sqrt{1-0}}$

$= \frac{1}{2} + \frac{1}{2} = 1$

$\lim_{x \to 0} \frac{\sqrt{1+x} - \sqrt{1-x}}{x} = 1$

Taylor Series :-

when a function defined on (a, b) and for $a < c < b$ all derivatives f^n exist on (a, b) and bounded by a single constant C then

$$\lim_{n\to\infty} R_n(x) = 0 \text{ for all } x \in (a, b)$$

i.e $\lim_{n\to\infty} \cancel{R_n(x)} = (or)$

$$0 = f(x) - \sum_{k=0}^{n-1} \frac{f^k(c)}{k!}(x-c)^k$$

Taylor Series at (x_0)

Let f be a function defined on some open Interval containing $x \in R$ and if f has derivatives of all order at $x_0 \Rightarrow$

$$\cancel{f(x)} = \sum_{k=0}^{\infty} \frac{f^{(k)}(x)(x-x_0)^k}{k!}$$ is called

Taylor Series of f about x_0

or.

$$\sum_{k=0}^{\infty} \frac{f^k(c)}{k!}(x-c)^k = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \dots$$

at $c = 0$

$$\sum_{k=0}^{\infty} \frac{f^k(0)}{k!} x^k = f(0) + \frac{f'(0)}{1!} + \frac{f''(0)}{2!} x^2 + \dots$$

(11) Find Taylor Series for $\cos x$.

given function

$\cos x \Rightarrow f(x) = \cos x \quad x \in R$.

derivatives of $f(x) = \cos x$ is

$$f'(x) = -\sin x$$
$$f''(x) = -\cos x \Rightarrow f^n(x) = \begin{cases} -\sin x & n = 1, 5, 9 \\ -\cos x & n = 2, 6, 10 \\ \sin x & n = 3, 7, 11 \ldots \\ \cos x & n = 0, 4, 8, 12 \ldots \end{cases}$$
$$f'''(x) = \sin x$$
$$f^{iv}(x) = \cos x$$

At $x = 0$, $\begin{cases} 0 \\ -1 \\ 0 \\ 1 \end{cases}$

$$f^n(0) = \begin{cases} (-1)^{2k+1} \sin 0 & n = 2k+1 \quad k = 0, 1, 2, 3, \ldots \quad k = 2(k)+1 \\ (-1)^k \cos 0 & n = 2k \end{cases}$$

or

$$f^n(0) = \begin{cases} 0 & n = 2k+1 \\ (-1)^k & n = 2k \end{cases} \quad \text{— I}$$

$k = 1, 2, 3, 4$.

so $|f^{(n)}(0)| < 1$ so bounded by constant 1 $\in (-\infty, \infty)$ and all derivates f^n exist on $(-\infty, \infty)$

$\Rightarrow$ Remainder $R_n(x)$ is defined by

$$R_n(x) = f(x) - \sum_{K=0}^{n-1} \frac{f^K(c)}{K!}(x-c)^K \quad \text{—} (1)$$

$\therefore f$ is bounded by single constant

$$\Rightarrow \lim_{n\to\infty} R_n(x) = 0 \quad \text{—} (2)$$

Sub (2) in (1)

$$\Rightarrow 0 = f(x) - \sum_{K=0}^{n-1} \frac{f^K(c)}{K!}(x-c)^K$$

$$f(x) = \sum_{K=0}^{n-1} \frac{f^K(c)}{K!}(x-c)^K$$

Sub $c=0$ in above equation

$$\Rightarrow f(x) = \sum_{K=0}^{n-1} \frac{f^K(0)}{K!} x^K \text{ but } \sum_{K=0}^{n-1} \frac{f^K(0)}{K!} x^{2K}$$ from I

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$\therefore$ Taylor series for $\cos x$ is

$$1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\cos x = \sum_{K=0}^{\infty} \frac{x^{2K}}{(2K)!}$$

(1) find Taylor Series for $\sin x$ for all $x \in R$

given $f(x) = \sin x$ for all $x \in R$.

derivative of $f(x) = \sin x$ is

$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f^{iv}(x) = \sin x.$$

generally

$$f^n(x) = \begin{cases} \cos x & n = 1, 5, 9, \\ -\sin x & n = 2, 6, 10, \ldots \\ -\cos x & n = 3, 7, 11, \ldots \\ \sin x & n = 0, 4, 8, 12 \ldots \end{cases}$$

at $x = 0$

$$f^n(0) = \begin{cases} (-1)^{2k+1} \\ 0 \end{cases} \qquad \begin{cases} 1 \\ 0 \\ -1 \\ 0 \end{cases}$$

or

~~$f^n(0)$~~

$$f^n(0) = \begin{cases} +1 & n = 1, 5, 9 \\ -1 & n = 3, 7, 11 \ldots \\ 0 & \text{otherwise} \end{cases}$$

Taylor series for sin n is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{k+1}$$

or

$|f^n(0)| \leq 1$ bounded by $-1 \in (-\infty, \infty)$

so

$$\lim_{n \to \infty} R_n(x) = 0$$

$$\Rightarrow \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

sin n =

$$\cancel{f(x) = f(0) + \frac{f'(0)}{1!} x^1} + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3$$

$$\sin x = 0 + \frac{1}{1!} x^1 + 0 - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \frac{1}{7!} x^7$$

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} \quad \forall x \in R.$$

Taylor Series for $\sinh x = \frac{1}{2}(e^{x} - e^{-x})$

$$\sinh x = \frac{1}{2}(e^{x} - e^{-x})$$

$$\frac{d}{dx} \sinh x = \cosh x$$

$$\frac{d}{dx} \cosh x = \sinh x$$

$\vdots$

$\therefore$ functions are derivatives of each other we get $\sinh x$, $\cosh x$ alternately.

$\therefore$ By Taylor's Theorem

$$\sum_{K=0}^{\infty} \frac{f^{K}(c)}{K!}(x-c)^{K} = f(c) + \frac{f'(c)}{1!}(x-c) + \dots$$

at $c = 0$

$$\sum_{n=0}^{n} \frac{f^{n}(0)}{n!} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^{2} + \dots$$

$$= \sinh(0) + \frac{x}{1!}\cosh(0) + \frac{x^{2}}{2!}\sinh(0) + x^{3}$$

$$= 0 + \frac{x^{1}}{1!} + 0 + \frac{x^{3}}{3!} + \frac{x^{5}}{5!} + \frac{x^{7}}{7!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

Date
Page

find Taylor Series for $\cosh x = \frac{1}{2}(e^{x} + e^{-x})$

$$f(x) = \cosh x = \frac{1}{2}\left(e^{x} + e^{-x}\right)$$

derivatives are, $\frac{d}{dx}\cosh x = \sinh x \Rightarrow \frac{d^2}{dx^2}(\cosh x) = \sinh x$

alternately sinh x, cosh x as derivative.

By Taylor's Theorem

$$\sum_{k=0}^{\infty} \frac{f^{k}(c)}{k!}(x-c)^{k} = f(c) + \frac{f'(c)}{1!}(x-c) + \dots$$

at $c = 0$

$$\sum_{k=0}^{\infty} \frac{f^{k}(0)}{k!}x^{k} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots$$

$$= \cosh(0) + \frac{x^1}{1!}\sinh(0) + \frac{x^2}{2!}\cosh(0) + \dots$$

$$= 1 + 0 + \frac{x^2}{2!} + 0 + \frac{x^4}{4!} + \frac{x^6}{6!}$$

$$= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!}$$

$$= \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$$

33

(13) ST $\sin x \le x$ for all $x \ge 0$

let

$f(x) = x - \sin x$ for $x \in (0, \infty)$

(13)

$\Rightarrow$ $f(0) = 0 - \sin 0 = 0$

The derivative of f is

$f'(x) = 1 - \cos x$

$= 1 - (1 - 2\sin^2 \frac{x}{2})$

$= 2\sin^2 \frac{x}{2} > 0$ for $x > 0$

$f'(x) > 0$ for $x > 0$

$\therefore$ The function f is increasing on $(0, \infty)$

$\Rightarrow$ for $x > 0$ $f(x) > f(0)$ because f is increasing function.

$x - \sin x > 0 - \sin 0$ — (1)

$x - \sin x > 0$

$x > \sin x$ for $x > 0$

if $x = 0$

$\sin x = \sin 0 = 0$

$\sin x = x$ for $x = 0$ — (2)

from (1) & (2) $x \ge \sin x$ or $\sin x \le x$ $\forall x \in [0, \infty)$ or $x \ge 0$

P.T. $|\cos x - \cos y| \le |x-y|$ for all $x, y \in R$

Let $f(t) = \cos t$ on R.
$\cos t$ is continuous on $[a,b]$ &
differentiable on (a,b)

$\Rightarrow f'(t) = -\sin t$
$\Rightarrow$ by mean value theorem $\exists$ a $c \in (a,b)$

$$f'(c) = \frac{f(x) - f(y)}{x-y}.$$

$$-\sin c = \frac{\cos x - \cos y}{x-y}$$

$$|-\sin c| = \left|\frac{\cos x - \cos y}{x-y}\right| \quad [|-\sin c| = |\sin c|]$$

but $0 \le |\sin c| \le 1$

So $|\sin c| \le 1$

$$\left|\frac{\cos x - \cos y}{x-y}\right| \le 1$$

$|\cos x - \cos y| \le |x-y|$ for $x, y \in R$

Hence proved.

(12) let $f(x) = x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$

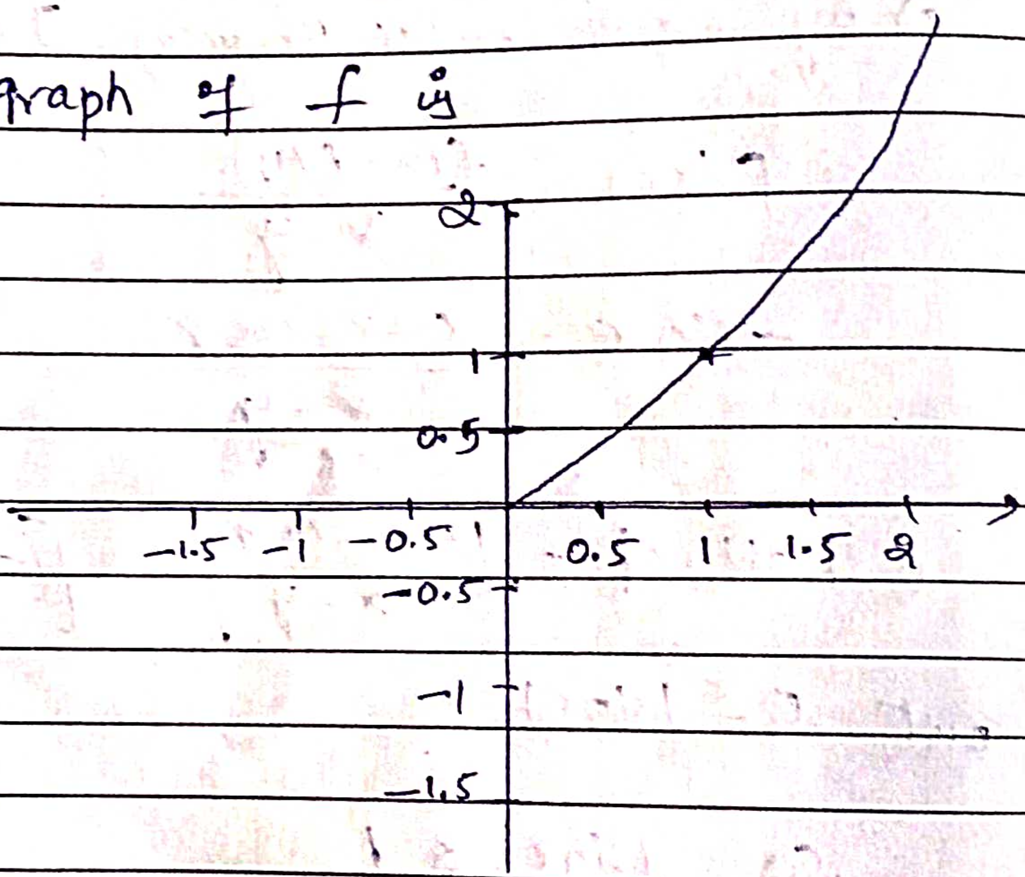
(12)

(i) sketch graph of f

(ii) check if f is differentiable at $x = 0$

give $f(x) = \begin{cases} x^2 & x \geq 0 \\ 0 ; & x < 0 \end{cases}$

graph of f is



Date
Page

differentiability $\quad x = a+h \quad x = a-h$

RHL $\quad$ LHL.

$$\lim_{h \to 0^+} \frac{f(a+h) - f(h)}{h} \quad \Bigg| \quad \lim_{h \to 0^-} \frac{f(a-h) - f(h)}{-h}$$

here $a = 0$ $\quad\quad a = 0$

$f(h) - f(h)$

differentiability for $a = 0$

$$\lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \frac{x^2 - 0}{x - 0} = \lim_{x \to 0} x = 0$$

$x \geq 0$

$$\lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \frac{0 - 0}{x - 0} = 0$$

$x < 0$

$\therefore f'(0) = 0$

f is differentiable at a

Verify Rolles Theorem in $[-2, 2]$ for

$$f(x) = x^2 - 1$$

(i) given $f(x) = x^3 - 1$

we know every polynomial is continuos and differentiable

(i) $f(x)$ is continuos on $[-2\ 2]$

(ii) $f(x)$ is differentiab in $(-2, 2)$

(iii) $f(-2) = (-2)^2 - 1 = 4 - 1 = 3$

$f(2) = (2)^2 - 1 = 4 - 1 = 3$

$\Rightarrow$ as f is continuos $[-2\ 2]$ $f(x)$ different on $(-2, 2)$ &

$$f(-2) = f(2)$$

So according to roll's Theorem

$\exists$ atleast one $c \in [-2, 2]$

where $f'(c) = 0$

$\Rightarrow$ $f(x) = x^3 - 1$

$f'(x) = 3x^2$

$f'(c) = 3c^2 \Rightarrow f'(c) = 0$

$3c^2 = 0 \Rightarrow c = 0$

$c = 0 \Rightarrow 0 \in [-2\ 2]$

Hence verified.