1 Real Analysis degree 3rd sem

UNIT-2

limits of function

continuos functions

uniform continuity

Continuity

properties of continuous functions.

Incrasing

bunded

Unit 2

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(2). SAS LAQ

1. Define continuos & uniform continuos.
2. If f is continuous at x_0 and g is continuos at $f(x_0)$ then composite function gof is continuos at x_0

3. Let f & g be real valued function continuos at x_0 if $g(x_0) \neq 0$ ⇒ PT

properties

(i) $f+g$ is continuos at x_0

(ii) fg is " at x_0

(iii) f/g is " " x_0 if $g(x_0) \neq 0$

2022 LAQ

4. If f is continuos on closed interval $[a, b]$ ⇒ f is uniformly continuosly on $[a,b]$

2022 LAQ

5. If f is continuos function on $[a, b]$ ⇒ PT f is bounded.

6. Define limit of function find $\lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x-a}$

7. PT $f(x) = [x]$ at $x=2$ is continuon on R by using $\epsilon - \delta$ definition.

Same for $f(x) = 2x^2+1$

unit-2

(3) LAQs

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1. Let f be continuos ~~and no and g~~ one to one function is continuos at Interval I then PT f is strictly Increasing.

2. State and prove continuos functions and [Intermediate value Theorem]

3. If $f(x)=\sqrt{4-x}$ for $x<4$ and $g(x)=x^2$ for all $x \in R$
   (a) give domains of $f+g$, fg, fog and gof
   (b) find values of $fog(0)$, $fog(1)$ $gof(2)$
   (c) Are functions fog and gof equal
   (d) Are $fog(3)$ and $gof(3)$ meaningful.

4. let $f(x)=\frac{\sqrt{1+3x^2}-1}{x^2}$ for $x\neq 0$ ST $\lim_{x\to} f(x)$ exists and determine its values

5. $f:R\to R$ be a function as $f(x)=\begin{cases} x^2\sin\frac{1}{x} & x\neq 0 \\ 0, & x=0\end{cases}$
   $\Rightarrow$ ST f is continuos at $x=0$

6. ST $f(x)=\frac{1}{x^2}$ is uniformly continuos on $[a, \infty]$ where $a>0$, $a\in R$.

7. f real valued function $dom(f)\subset R$. f is continues at x_0 in $dom(f)$ if & only if for each $\varepsilon>0$ $\exists$ $\delta>0$ ~~x~~ $|x-x_0|<\delta \Rightarrow |f(x)-f(x_0)|<\varepsilon$

Basics (4)

→ ~~Define a continuos function~~

function :- $f(x) = 5x$

$x=1 \Rightarrow f(1) = 5(1) = 5$

$x=3 \Rightarrow f(3) = 5(3) = 15$

$\Rightarrow \{1, 3\} \xrightarrow{5x} 5, 15$

function

I/p — operation — o/p

domain — codomain, Range

list of values by a formula) i/p's & o/p's

Sequence	function ~~limit~~:	limit
$\{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$	$f(x) = \frac{1}{x}$: i/p → x	$f(n) = \frac{1}{n}$
$s_1, s_2, s_3, \ldots$	o/p → $\frac{1}{x}$	$\lim_{n\to 1} f(n) = 1$
$\Rightarrow s_n = \frac{1}{n}$ $n \geq 1$	$f(1) = 1$	$\lim_{n\to 2} f(2) = \frac{1}{2}$
Converging :- close to a speific value. i.e limit	$f(2) = 1/2, \ldots$	$\lim_{n\to\alpha} f(n) = \frac{1}{\alpha} = 0$
$f(n) = \frac{1}{n}$		The value approached by a sequence/function as a input or index approaches a specific value.
$f(1) = 1$		
$f(2) = \frac{1}{2} = 0.5$		
$f(3) = \frac{1}{3} = 0.25$		
Converges to zero		
$\lim_{n\to\alpha} f(n) = \lim_{n\to\alpha} \frac{1}{n} = 0$		
→ it : wont oscillate or diverge.		

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Basics

Continuos function :-

function as a graph ⟹ (1) No gaps/hole
is continuos if, (2) No Jumps/discontinuos
⟶ (3) Smooth unbroke curve

Continuos ⇒ :- function $f(x) = x^2$ for any value of x

(i) A defined output value.
(ii) A limit that exists as x-approaches any value
(iii) the limit equal the function value

$[5, \to 8]$ $f(x)$ subse $\{6,7\}$ point subset set

SA 1 Continuos function

let f be a function

(i) given $x_0 \in$ domain(f) f is said to be continuos at x_0 if for any sequence (x_n) in domain(f) converging to x_0

$$\lim_{n\to\infty} f(x_n) = f(x_0)$$

(ii) f is continuos at each point of subset $S \subset$ domain$(f) \Rightarrow$ f is continuos on S

(iii) f is said to be continuos if f is continuos on dom(f)

Pdf an Questions order

SA-1

SA-7

SA-6

SA-2

SA-3

LA-3

LA-5

SA-7

SA-5

SA-4

LA 2

LA-1

LA-4

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Q(7) (2) $f(x) = 2x^2+1$ ST f is continuos by definition & by using $\epsilon - \delta$ properfery.
E-S

Definition :- $f(x) = 2x^2+1$ for $x \in R$.

$\Rightarrow$ let x_n be a sequence &

$\lim x_n = x_0$

$$\lim f(x_n) = \lim (2x_n^2+1)$$
$$= 2(\lim x_n)^2+1$$
$$= 2x_0^2+1$$
$$= f(x_0)$$

$$\lim f(x_n) = f(x_0)$$

$\therefore$ f is continuos at each x_0 in R.

by using $\epsilon - \delta$ property :-

Cauchy's definition of continuity.

A real values function f defined on a Interval I said to be continuos at $x = a \in I$. for each $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ when $|x-a| < \delta$

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let x_0 be in $\mathbb{R}$ and let $\epsilon > 0$

$\Rightarrow$ we need to show $|f(x) - f(x_0)| < \varepsilon$

$$|f(x) - f(x_0)| = |2x^2 + 1 - (2x_0^2 + 1)|$$
$$= |2x^2 - 2x_0^2|$$
$$= |2(x^2 - x_0^2)|$$
$$= 2\,|x - x_0|\,|x + x_0|$$

here $|x + x_0|$ should be bounded

if $|x - x_0| < 1$

$\Rightarrow$ $|x| < |x_0| + 1$

$$|x + x_0| \leq |x| + |x_0| < 2|x_0| + 1$$

$\Rightarrow$ $|f(x) - f(x_0)| < 2(2|x_0| + 1) \cdot |x - x_0|$ if $|x - x_0| < 1$

$\Rightarrow$ $\delta = \min\left\{1, \dfrac{\varepsilon}{2(2|x_0| + 1)}\right\}$

$\Rightarrow$ $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

$\therefore$ f is continuos.

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③ ⑥ Define limit of a function and find

SA 6

$$\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a}$$

limit of a function :-

limit of a function 'f' defined on S denoted by

$$\lim_{x \to a}{}_{s} f(x) = L.$$

S is subset of R

a Real number or symbol $+\infty$ or $-\infty$

L is Real number or symbol $+\infty$ or $-\infty$

for every sequence (x_n) in S with limit a

$$\lim_{n\to\infty} f(x_n) = L.$$

(ii) $\lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{x-a} = \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x})^2-(\sqrt{a})^2}$

$$= \lim_{x\to a} \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})}$$

$$= \lim_{x\to a} \frac{1}{\sqrt{x}+\sqrt{a}} = \frac{1}{\sqrt{a}+\sqrt{a}} = \frac{1}{2\sqrt{a}}$$

$(a+b)(a-b) = a^2-b^2$

$a^3-b^3 = (a-b)(a^2+b^2+ab)$

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(7) f is continuos at x_0 and g is
SA (2) continuos at $f(x_0)$ $\Rightarrow$ PT composite
function $g \circ f$ is continuos at x_0

let x_0 be the Sequence.

$x_n \in \operatorname{dom} f \quad f(x_n) \in \operatorname{dom} g)$

$\&$ $\lim x_n = x_0$

given that f is continuos at x_0

$\Rightarrow \lim f(x_n) = f(x_0)$

g is continuos at $f(x_0)$

$\lim g \circ f(x_n) = \lim g(f(x_n))$

$= g \lim f(x_n))$

$= g(f(x_0))$

$= g \circ f(x_0)$

$\lim g \circ f(x_n) = g \circ f(x_0)$

$\therefore$ $g \circ f$ is continuos at x_0.

Same & Intersection

$\Rightarrow$ also work

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(5) let f and g be real valued function continuos at x_0

SA 3

PT (i) $f+g$ is continuos at x_0

(ii) fg is " x_0

(iii) f/g is " " x_0 if $g(x_0) \neq 0$

f and g are real values function

given $x_0 \in \text{dom} f$ $x_0 \in \text{dom} g$

and at x_0 f & g continuos.

$\Rightarrow \lim x_n = x_0$ $\quad \lim f(x_n) = f(x_0)$

$\lim g(x_n) = g(x_0)$

(i) $\lim (f+g)(x_n) = \lim (f(x_n) + g(x_n))$

$= \lim f(x_n) + \lim g(x_n)$

$= f(x_0) + g(x_0)$

$= f+g(x_0)$

$\therefore$ $f+g$ is continuos at x_0

(ii) $\lim fg(x_n) = \lim f(x_n) g(x_n)$

$= \lim f(x_n) \cdot \lim g(x_n)$

$= f(x_0) \cdot g(x_0)$

$= fg(x_0)$

$\therefore$ fg is continuos at x_0

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(iii) $\lim g(x) \neq 0$

$$\lim \left(\frac{f}{g}\right) x_n = \lim \frac{f(x_n)}{g(x_n)} = \frac{\lim f(x_n)}{\lim g(x_n)}$$

$$= \frac{f(x_0)}{g(x_0)}$$

$$\lim \left(\frac{f}{g}\right) x_n = \frac{f}{g}(x_0)$$

$\therefore$ f/g is is continuos at x_0

LA 3

(6) Let $f(x) = \sqrt{4-x}$ for $x \le 4$, $g(x) = x^2$ for all $x \in R$.

(a) give domains of $f+g$, fg and $f \circ g$, $g \circ f$

(a) $f(x) = \sqrt{4-x}$ for $x \le 4$
$g(x) = x^2$ for all $x \in R$.

(i) $f+g = \sqrt{4-x} + x^2$ has domain $\{x : x \le 4\}$
(ii) $fg = x^2\sqrt{4-x}$ has domain $\{x : x \le 4\}$

(iii) $f \circ g = f(g(x)) = f(x^2) = \sqrt{4-x^2}$

The domain of $f \circ g$ is set of x with x in domain of g
and $g(x)$ in domain of f
$x \in 4$ and $x^2 \le 4$

$\therefore x \in [-2, 2]$

(iv) $g \circ f = g(f(x)) = g(\sqrt{4-x})^2 = (\sqrt{4-x})^2 = 4-x$
The domain of $g \circ f$ is the set of x where $x \le 4$
and $\sqrt{4-x} \in R$.
$\therefore$ domain is $\{x : x \le 4\}$

$f(x) = \sqrt{4-x} \qquad g(x) = x^2$

(b) (i) $f\circ g(0) = f(g(0)) = f(0) = \sqrt{4-0} = 2$

(ii) $g\circ f(0) = g(f(0)) = g(\sqrt{4}) = g(2) = 2^2 = 4$

(iii) $f\circ g(1) = f(g(1)) = f(1) = \sqrt{4-1} = \sqrt{3}$

(iv) $g\circ f(1) = g(f(1)) = g(\sqrt{3}) = (\sqrt{3})^2 = 3$

(v) $f\circ g(2) = f(g(2)) = f(2^2) = f(4) = 0$

(vi) $g\circ f(2) = g(f(2)) = g(\sqrt{2}) = \sqrt{2}^2 = 2$

(c) $f\circ g = f(g(x)) = f(x^2) = \sqrt{4-x^2}$

$g\circ f = g(f(x)) = g(\sqrt{4-x}) = 4-x$

$f\circ g \neq g\circ f$. both are not equal

(d) The domain of $f\circ g$ is $[-2, 2]$ so

$f\circ g(3)$ does not make sense.

The domain of $g\circ f$ is $x \leq 4$

i.e. $(-\infty, 4]$

$g\circ f(3) = g(f(3)) = g(1) = 1^2 = 1$

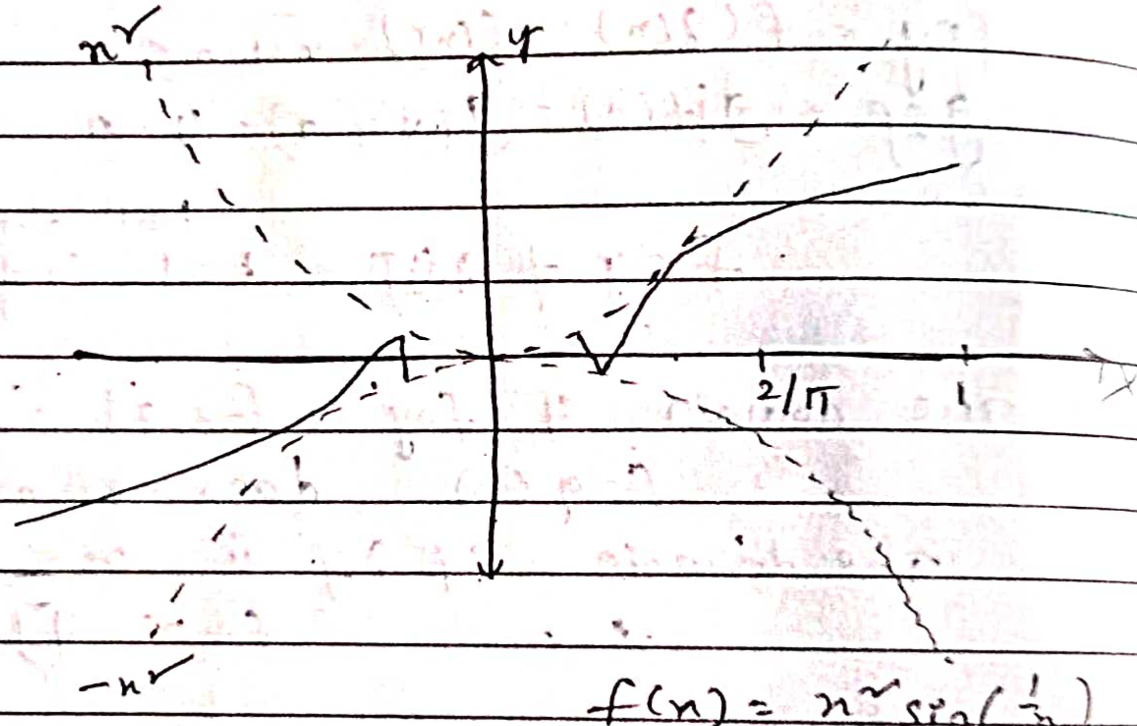
⑦ $f: R \to R$ be a function as

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Ans

$\Rightarrow$ ST f is continuos at $x = 0$

given $\begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$



let $\varepsilon > 0$

$$|f(x) - f(0)| = |x^2 \sin\tfrac{1}{x} - 0|$$
$$= |x^2 \sin\tfrac{1}{x}|$$
$$= x^2 |\sin(\tfrac{1}{x})|$$
$$|f(x) - f(0)| \leq x^2$$

let $\delta = \sqrt{\varepsilon}$ $\Rightarrow$

if $|x-0| < \delta$ and $x^2 < \delta^2 = \varepsilon$

$$|f(x) - f(0)| \le x^2 < \varepsilon .$$

$\therefore$ f is continuos at x_0.

Similar Q.

(1) if $\sin x, \cos x, e^x, x^n \log_e x$ are continuos
$\Rightarrow$ PT following functions are continuos

(i) $\log_e(1+\cos^4 x)$ $\cos x \to \cos x$ $fg \to$

$\cos x$ is continuos $\Rightarrow$ $\cos^4 x$ is continuos [f & g continuos $\Rightarrow$ fg continuos]

$y = 1$ continous

$1+\cos^4 x$ continus [f+g continuos]

$\log x$ is continuos for $x > 0$

$\log(1+\cos^4 x)$ is continuos.

(ii) $(\sin x^2 + \cos^6 x)^\pi$ ~~are~~ continuos

$\sin x$ & $\cos x$ are continuos

$\Rightarrow$ $\sin^2 x$ & $\cos^6 x$ continuos

$f(x) = \sin^2 x + \cos^6 x$ continuos

$g(x) = x^\pi$ continuos for $x > 0$

$\Rightarrow$ $\cancel{fog(x)}$ $gof(x) = (\sin^2 x + \cos^6 x)^\pi$ is continuos for $\forall x \in R$.

2^{x^2}

(c) $f(x) = 2^x$, $g(x) = x^2$

$f(x)$ & $g(x)$ continuos for all x.

$\Rightarrow$ $fog(x) = 2^{x^2}$ is continuos

(d) $8^x = 2^{3x} = (2^x)^3$

let $f(x) = 2^x$ and $g(x) = x^3$

f and g continuos for all x.

$\therefore$ $gof(x) = g(f(x)) = g(2^x) = 2^{x^3} = 8^x$

is also continuos. $\forall x \in R$.

properties of continuos functions

(i) f & g are continuos for $x \in R$.

$\Rightarrow$ (ii) $f+g$ is also continous $\forall x \in R$

(ii) fg " " " $\forall x \in R$

(iii) f/g is also continus $g \neq 0$

(iv) $fog(x)$

(v) $gof(x)$

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SA 7 PT $f(x) = x^2$, $x_0 = 2$. Using $\epsilon - \delta$ property

$f(x) = x^2$, $x_0 = 2 \Rightarrow f(x_0) = 2^2 = 4$

By definition of continuity $\lim_{x \to x_0} f(x) = f(x_0)$

$$\lim_{x \to x_0} f(x) = \lim_{x \to 2} x^2$$

$$= 2^2 = 4 = f(x_0)$$

$$= f(2)$$

$$\lim_{x \to 2} f(x) = f(2)$$

$\therefore$ $f(x) = x^2$ is continuos at $x_0 = 2$.

$\epsilon - \delta$ property.

for every $\epsilon > 0$ $\exists$ a $\delta > 0$
Such that $|f(x) - f(x_0)| < \epsilon$
when $|x - x_0| < \delta$

Here $x_0 = 2$.

consider

$$|f(x) - f(2)| = |x^2 - 2^2|$$

$$= |x-2| . |x+2|$$

$$= |x-2| . (|x| + |2|)$$

The bound for $|x+2|$ That does not depend on x if $|x-2| < \delta$

Then
$$||x|-|2|| \le |x-2|$$
$$|x|-|2| < \delta$$
$$|x| < |2| + \delta$$

So,
$$|x+2| \le |x|+|2|$$
$$< |2| + \delta + |2|$$
$$\cancel{\le}\ 4+\delta$$
$$\therefore\ |f(x)-f(2)| \le |x-2|\,(4+\delta)$$
choose $\delta = 1$
$$|f(x)-f(2)| \le |x-2|\,5$$
Thus, $5|x-2| < \varepsilon$
$$|x-2| < \frac{\varepsilon}{5}$$

Now set $\delta = \min\{1, \frac{\varepsilon}{5}\}$

$\Rightarrow$ $|x-2| < \delta$ implies

$$|f(x)-f(2)| < \varepsilon$$

$\therefore$ $f(x) = x^2$ is continuos at $x_0 = 2$

(8) if :- f is bounded function ;

SA 5 f is continuos function on closed

Imt interval $[a, b]$ then prove that f

LAQ is bounded.

Bounded function :-

f be a real valued function f is said to be bounded if $\{f(x) : x \in \text{dom} f\}$ is a bounded set

if there exists $M \in \mathbb{R}$ such that

$|f(x)| \leq M$ for all $x \in \text{dom} f$

f is continuos $\Rightarrow$ f is bounded :-

let f is not bounded on $[a, b]$

Then for $n \in \mathbb{N}$ the set of values

$\{|f(x)| : x \in [a, b]\}$ is not bound by n

considering for all $n \in \mathbb{N}$ we obtain sequence

x_n, $x_n \in [a, b]$ such that

$|f(x_n)| > n$.

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By Bolzano weierstrass Theorem (x_n) has a Subsequence (x_{nk}) such that,

$\lim x_{nk} = x_0$.

Since $a \leq x_n \leq b$ we have $x_0 \in [a,b]$
by continuos of f on $[a\ b]$

$$\lim_{k\to\infty} f(x_{nk}) = f(x_0)$$

This is a contradiction to

$$\lim_{k\to\infty} |f(x_{nk})| = +\infty$$

Contradiction Rised due to our assumption
$\therefore$ f is not bounded.

$\therefore$ our Assumption was wrong

$\therefore f$ is bounded.

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(9) If f is continuos on closed interval [a b] PT f is uniformly continuos on [a, b]

SA 4 but LAQ

Uniform contintyity :- let $f(x)$ be a real valued function $S \subset R$.

$\Rightarrow$ f is uniform continuos on S if each $\varepsilon > 0$ There exists a $\delta > 0$

$x, y \in S$ $|x - y| < \delta$

$|f(x) - f(y)| < \varepsilon$

RTP :- $\boxed{f \text{ is a UC}}$

proof :-

$f \rightarrow$ continuos fuction

let f is not uniform contiful

for each $\varepsilon > 0$ for each $\delta > 0$

$\left.\begin{array}{l} |x - y| < \delta \\ |f(x) - f(y)| < \varepsilon \end{array}\right\}$ not applicable fail

$\delta < 0 \Rightarrow x, y \in S$

$|x - y| < \delta \Rightarrow f(x) - f(y) \geq \varepsilon$

for each $n \in N$ x_n, y_n on [a b]

$|x_n - y_n| < \frac{1}{n}$ and

$|f(x_n) - f(y_n)| \geq \varepsilon$

for Bolzano weirstrass Theorem

Sequence x_{n_k} converges to x_0

$$\lim_{k \to \infty} x_{n_k} = x_0 \qquad x_0 \in [a, b]$$

$$\lim_{k \to \infty} y_{n_k} = x_0$$

f is continuos

$$f(x_0) = \lim_{k \to \infty} f(x_{n_k}) = \lim_{k \to \infty} f(y_{n_k})$$

$$\lim_{k \to \infty} [f(x_{n_k}) - f(y_{n_k})] = 0$$

$$|f(x) - f(y)| \geq \varepsilon \qquad \varepsilon > 0$$

$$|f(x_{n_k}) - f(y_{n_k})| \geq \varepsilon$$

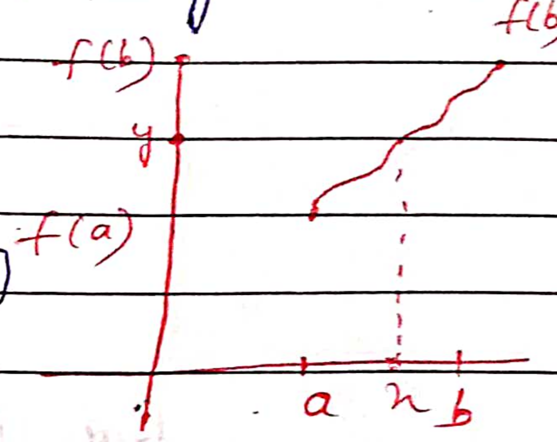
$$\lim_{k \to \infty} f(x_{n_k}) - f(y_{n_k}) = 0$$ — Is a contradiction

f is uniformly continuous Ly

(1) S.T $f(x) = \frac{1}{x^2}$ is uniformly continuos on $[a, \infty)$ where $a > 0$ $a \in R$

(10) State and prove Intermediate Value Theorem for Continuos functions.

Statement:- If f is a continuos function on an Interval I i.e $[a, b]$ and when $a, b \in I$ $a<b$ and $f(a) < y < f(b)$, $\exists$ at least one $x \in a, b$ such that $f(x) = y$.



proof:- given that f is Continuos function on $[a\ b]$

$a < b$

and $f(a) < y < f(b)$ — (1)

let us consider a function

$\phi(x) = f(x) - y$ =>

$\cancel{\phi(a)}$

Since f is a continuos function then $\phi(x)$ is also a continuos founction. on $[a\ b]$

$f(a) < y < f(b)$

=> $\phi(a) = f(a) - y$ | $f(a) < y$ | $f(b) > y$

$\phi(b) = f(b) - y$ | $\phi(a) + y < y$ | $\phi(b) + y > y$

we have $a < b$ on $[a\ b]$ $\phi(a) < 0$

$\phi(b) > 0$

Then

$$\phi(a) < \phi(b)$$

$$\phi(a) < \phi(x) < \phi(b) \quad x \in [a\ b]$$

$$f(a)-y < f(x)-y < f(b)-y.$$

$$f(a)-y+y < f(x)-y+y < f(b)$$

$f(a) < f(x) < f(b)$ for at least one $x \in [a\ b]$ ②.

from ①: $f(a) < y < f(b)$.

$\therefore$ from ① & ②

There exist at least one $x \in [a, b]$ for which

$$f(x) = y.$$

$[a\ b]$
$\phi(a) < 0$
$\phi(b) > 0$
$x \in (a\ b)$
$\phi(x) = 0$
$f(x) - y = 0$
$f(x) = y$

$\therefore$ Intermediate value Theorem proved.

If f is continuous on $[a\ b]$ & $f(a), f(b)$ have opposite signs $\Rightarrow \exists$ a $c \in (a, b)$ $f(c) = 0$

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(11) Let f be continuos at x_0 and
~~g is continuos at~~ one to one function Intervel J
LA-1 Then f is continu strictly Increasing

Strictly Increasing function :-
Real valued fuction $f \Rightarrow I$ is said to f
f is strictly increasing on I
if $x_1 \& x_2 \in I$ $x_1 < x_2$ $\rightarrow$ $\rightarrow$
and $f(x_1) < f(x_2)$ $\rightarrow$
$\Rightarrow$ f is strictly Increasing function

given,
f continuos on I, one to one
$x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$
$x_1, x_2 \in I$, $x_1 < x_2$

Case(i) $\Rightarrow$ $x_1 < x_2$ for $f(x_1) = f(x_2)$
f is one to fuction $x_1 = x_2$ — (1)
but contradicts $x_1 < x_2$
$\therefore x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

Case-(ii)
$f(x_1) > f(x_2)$

$g(x) = f(x) - f(x_2) \Rightarrow g(x)$ is continuos
By Intermadiate value Theorem | $g(x_1) > 0$

when f is continuos on $[a\ b]$

and $a < b$ = & $f(a) < y < f(b)$

$\exists$ at leastone $x \in [a, b]$

$f(x) = y$. $\Rightarrow f(x) - y = 0$

here g is continuos on I.

$x_1 < x_2$ & $\exists$ $x \in I$ i.e $x_1 < x_3 < x_2$

$g(x_3) = 0$

$f(x) - f(x_2) = 0$

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(12) Let $f(x) = \frac{\sqrt{1+3x^2}-1}{x^2}$ for $x \neq 0$

LA4

ST $\lim_{x \to 0} f(x)$ exists and determines its value

given $f(x) = \frac{\sqrt{1+3x^2}-1}{x^2}$ for $x \neq 0$

$\Rightarrow \lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \frac{\sqrt{1+3x^2}-1}{x^2}$

$= \lim_{h \to 0} \frac{\sqrt{1+3(0-h)^2}-1}{(0-h)^2}$ $\left[\begin{array}{l} x \to 0^- \\ h \to 0 \\ x = 0-h \end{array} \right.$

$= \lim_{h \to 0} \frac{\sqrt{1+3h^2}-1}{h^2}$

Rationalizing numerator:

(13) $f(x) =$