unit-2 Cones

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① find eqn of cone whose vertex is (α,β,γ) and base $y^2=4ax, z=0$

given vertex is (α,β,γ) base eqn is
$y^2=4ax,\ z=0$.

$\Rightarrow$

Eqn of line passing through (α,β,γ)
is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ — ②

given that $z=0 \Rightarrow \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}=\frac{-\gamma}{n}$

$\frac{x-\alpha}{l}=\frac{-\gamma}{n}$ & $\frac{y-\beta}{m}=\frac{-\gamma}{n}$

$x-\alpha=\frac{-l\gamma}{n}$, $y=\frac{-\gamma m}{n}+\beta$

$x=\alpha-\frac{l\gamma}{n}$ ~~here also~~ $\left(\cancel{y-\beta}=\frac{-\gamma}{n}\right)$

$\therefore$

$(\because x,y,z)=\left(\alpha-\frac{l\gamma}{n},\ \beta-\frac{\gamma m}{n},\ 0\right)$

The point lies on $y^2=4ax$.

$\Rightarrow \left(\beta-\frac{\gamma m}{n}\right)^2=4a\left(\alpha-\frac{l\gamma}{n}\right)$ — ③

$\cancel{\left(\beta-(y-\beta)\right)^2=4a\left(\alpha-(x-\alpha)\right)}$.

from eqn ② $\frac{y-\beta}{m}=\frac{z-\gamma}{n} \Rightarrow \frac{y-\beta}{z-\gamma}=\frac{m}{n}$
$\frac{x-\alpha}{l}=\frac{z-\gamma}{n} \Rightarrow \frac{x-\alpha}{z-\gamma}=\frac{l}{n}$ — ④

Sub ④ in ③

$$\left[\beta-\gamma\left(\frac{y-\beta}{z-\gamma}\right)\right]^2=4a\left(\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right)$$

$$\left[\beta - \frac{\gamma-\beta}{(z-\gamma)}\gamma\right]^2 = 4a\left[\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right]$$

$$\left[\frac{\beta(z-\gamma)-(y-\beta)\gamma}{z-\gamma}\right]^2 = 4a\left[\frac{\alpha(z-\gamma)-(x-\alpha)\gamma}{z-\gamma}\right]$$

$$\frac{(\beta z - \cancel{\beta\gamma} - \gamma y + \cancel{\beta\gamma})^2}{(z-\gamma)^{\cancel{2}}} = 4a\left[\frac{\alpha z - \alpha\gamma - x\gamma + \alpha\gamma}{\cancel{z-\gamma}}\right]$$

$$(\beta z - \gamma y)^2 = 4a(\alpha z - \gamma x)(z-\gamma)$$

is the required equtiion of cone.

(2) find equition of cone whose vertex is point $(1,1,0)$ and whose guiding curve is $y=0$, $x^2+z^2=4$

given vertex is $(1,1,0)$ $y=0$
and guiding curve $x^2+z^2=4$ — (1)

eqn of line through $(1,1,0)$ is

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-0}{n}$$

$$\frac{x-1}{l} = \frac{y-1}{m} \text{ and } \frac{y-1}{m} = \frac{z}{n}$$

$y=0 \Rightarrow \quad \frac{x-1}{l} = \frac{-1}{m} \quad \& \quad \frac{-1}{m} = \frac{z}{n}$

$$x = -\frac{l}{m}+1 \qquad z = \frac{-n}{m}$$

$$\therefore (x,y,z) = \left(1-\frac{l}{m}, 0, \frac{-n}{m}\right)$$

Sub (x, y, z) in (1)

$$\left(1-\frac{l}{m}\right)^2+\left(\frac{-n}{m}\right)^2=4.$$

$$1+\frac{l^2}{m^2}-2\frac{l}{m}+\frac{n^2}{m^2}=4 \quad -(3)$$

from, $\cancel{\frac{x-1}{l}=\frac{-1}{m}=\frac{z}{n} \Rightarrow \frac{x-1}{-1}=\frac{l}{m}}$

$\cancel{\text{and } \frac{-1}{m}=\frac{z}{n} \Rightarrow \frac{n}{m}=-z}$

from $\frac{x-1}{l}=\frac{y-1}{m}=\frac{z}{n}$

$$\left.\begin{array}{l} \frac{x-1}{l}=\frac{y-1}{m} \Rightarrow \frac{l}{m}=\frac{x-1}{y-1} \ \& \\ \frac{y-1}{m}=\frac{z}{n} \Rightarrow \frac{n}{m}=\frac{z}{y-1} \end{array}\right\} (4)$$

Sub (4) in (3).

$$1+\frac{(x-1)^2}{(y-1)^2}-2\frac{x-1}{y-1}+\frac{z^2}{(y-1)^2}=4.$$

$$(y-1)^2+(x-1)^2-2(x-1)(y-1)+z^2=4(y-1)^2$$

$$y^2+1-2y+x^2+1-2x-2xy+2x+2y-2+z^2$$
$$=4y^2+4-8y.$$

$$x^2+y^2+z^2-2xy=4y^2+4-8y$$
$$x^2+y^2+z^2-2xy-4y^2-4+8y=0$$

$\therefore$ Cone is $x^2+y^2+z^2-2xy+8y-4=0$

Q1. Find the equation of the cone whose vertex is (α, β, γ) and base $ax^2 + by^2 = 1, z = 0$.

Answer :

Given,

Vertex = (α, β, γ)

And base equation is,

$ax^2 + by^2 = 1, z = 0$

Equation of line passing through (α, β, γ) is,

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(1)}$$

Since, it cuts the line $z = 0$ ...(2)

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

Equating the terms,

$$\frac{x-\alpha}{l} = \frac{-\gamma}{n}$$

$$\Rightarrow \quad x = \frac{-l\gamma}{n} + \alpha$$

$$\frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$$\Rightarrow \quad y = \frac{-m\gamma}{n} + \beta$$

$\therefore \quad (x, y, z) = \left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$ lies on equation (1),

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 = 1$$

$$\Rightarrow \quad a\left[\alpha - \frac{(x-\alpha)\gamma}{z-\gamma}\right]^2 + b\left[\beta - \frac{(y-\beta)\gamma}{z-\gamma}\right]^2 = 1 \qquad [\because \text{From equation (2)}]$$

$$\Rightarrow \quad a\left[\frac{\alpha(z-\gamma)-(x-\alpha)\gamma}{(z-\gamma)}\right]^2 + b\left[\frac{\beta(z-\gamma)-(y-\beta)\gamma}{(z-\gamma)}\right]^2 = 1$$

$$\Rightarrow \quad a[\alpha z - \alpha\gamma - x\gamma + \alpha\gamma]^2 + b[\beta z - \beta\gamma - y\gamma + \beta\gamma] = (z-\gamma)^2$$

$$\Rightarrow \quad a[\alpha z - x\gamma]^2 + b[\beta z - y\gamma]^2 = (z-\gamma)^2$$

$\therefore$ The required cone equation is,

$$a[\alpha z - x\gamma]^2 + b[\beta z - y\gamma]^2 = (z-\gamma)^2$$

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Reasons
cone in αβγ
⇒ P, x, y, z
only

P.I

Q2. Find the equation of the cone whose vertex is the origin and base curve is $z = 4, x^2 + y^2 - 48 = 0$.

Answer : (OU) June/July-22, Q4

Given,

Vertex is origin $O(0, 0, 0)$

Base curve is,

$$z = 4, x^2 + y^2 - 48 = 0 \qquad \text{...(1)}$$

Equation of line through (0, 0, 0) is,

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{...(2)}$$

Since it meets the line $z = 4$,

$$\frac{x}{l} = \frac{y}{m} = \frac{4}{n}$$

Equating the terms,

$$\frac{x}{l} = \frac{4}{n} \Rightarrow x = \frac{4l}{n}$$

$$\frac{y}{m} = \frac{4}{n} \Rightarrow y = \frac{4m}{n}$$

$\therefore \quad (x, y, z) = \left(\frac{4l}{n}, \frac{4m}{n}, 4\right)$ lies on equation (1).

$$\Rightarrow \quad \left(\frac{4l}{n}\right)^2 + \left(\frac{4m}{n}\right)^2 = 48$$

$$\Rightarrow \quad \frac{16l^2}{n^2} + \frac{16m^2}{n^2} = 48$$

$$\Rightarrow \quad \frac{l^2}{n^2} + \frac{m^2}{n^2} = \frac{48}{16}$$

$$\Rightarrow \quad \frac{l^2}{n^2} + \frac{m^2}{n^2} = 3$$

$$\Rightarrow \quad l^2 + m^2 = 3n^2$$

$\Rightarrow \quad x^2 + y^2 = 3z^2 \qquad [\because$ From equation (2)]

$\therefore$ The required cone equation is, $x^2 + y^2 - 3z^2 = 0$.

Q5. Find the enveloping cone of the sphere $x^2 + y^2 + z^2 + 2x - 2y = 2$ with its vertex at (1, 1, 1). (OU) July-21

vertex at (1, 1, 1)

$(A+B+C)^2 = A^2+B^2+C^2+2AB+2BC+2CA$

Answer :

Given,

Equation of sphere is,

$$S \equiv x^2 + y^2 + z^2 + 2x - 2y = 2$$

and vertex, $(\alpha, \beta, \gamma) = (1, 1, 1)$

$$S_1 = (1)^2 + (1)^2 + (1)^2 + 2(1) - 2(1) - 2$$

$\Rightarrow$ $S_1 = 1$

The equation of tangent plane is given as,

$$T \equiv \alpha x + \beta y + \gamma z + u(x + \alpha) + v(y + \beta) + w(z + \gamma) + d = 0$$

From equation (1),

$$u = 1, v = -1, w = 0, d = -2$$

$$T \equiv (1)x + (1)y + (1)z + 1(x + 1) - 1(y + 1) + 0(z + 1) - 2 = 0$$

$\Rightarrow$ $T \equiv x + y + z + x + 1 - y - 1 + 0 - 2 = 0$

$\Rightarrow$ $T \equiv 2x + z - 2 = 0$

The equation of the enveloping cone is given by, $SS_1 = T^2$

opening -,+ ध्यान दें।

i.e., $x^2 + y^2 + z^2 + 2x - 2y - 2(1) = (2x + z - 2)^2$

$\Rightarrow$ $x^2 + y^2 + z^2 + 2x - 2y - 2 = (2x)^2 + z^2 + (2)^2 + 2(2x)(z) + 2(z)(-2) + 2(2x)(-2)$

$\Rightarrow$ $x^2 + y^2 + z^2 + 2x - 2y - 2 = 4x^2 + z^2 + 4 + 4xz - 4z - 8x$

$\Rightarrow$ $x^2 + y^2 + z^2 + 2x - 2y - 2 - 4x^2 - z^2 - 4 - 4xz + 4z + 8x = 0$

$\Rightarrow$ $-3x^2 + y^2 + 10x - 2y - 6 - 4xz + 4z = 0$

$\Rightarrow$ $3x^2 - y^2 + 4xz - 10x + 2y - 4z + 6 = 0$

2ux+2uy [illegible] करें

$\therefore$ The required enveloping cone is,

pic

$$3x^2 - y^2 + 4xz - 10x + 2y - 4z + 6 = 0.$$

Q6. Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2x + 4z - 1 = 0$ with its vertex at (1, 1, 1).

vertex (1, 1, 1)

[(KU) PP-1, Q4 | (OU) Nov./Dec.-18, Q

OR

Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2x + 4z = 1$ with vertex at (1, 1, 1).

(KU) Dec.-18, Q1(

Answer :

Given,

Equation of sphere is,

$$S \equiv x^2 + y^2 + z^2 - 2x + 4z - 1 = 0 \quad \text{...(1)}$$

And vertex is (1, 1, 1)

2 ५ 2 [illegible]

$$S_1 = (1)^2 + (1)^2 + (1)^2 - 2(1) + 4(1) - 1$$

$\Rightarrow$ $S_1 = 4$

The equation of tangent plane from equation (1) is,

$$T = \alpha x + \beta x + \gamma z + u(x + \alpha) + v(y + \beta) + w(z + \gamma) + d = 0,$$

From equation (1),

$$u = -1, v = 0, w = 2, d = -1$$

$$T = (1)x + (1)y + (1)z - 1(x + 1) + 0 + 2(z + 1) - 1 = 0$$

$\Rightarrow$ $T = x + y + z - x - 1 + 2z + 2 - 1 = 0$

$\Rightarrow$ $T = y + 3z$

The equation of the enveloping cone is given by $SS_1 = T^2$

i.e., $(x^2 + y^2 + z^2 - 2x + 4z - 1)(4) = (y + 3z)^2$

$\Rightarrow$ $4x^2 + 4y^2 + 4z^2 - 8x + 16z - 4 = (y^2 + 9z^2 + 6yz)$

$\Rightarrow$ $4x^2 + 3y^2 - 5z^2 - 8x + 16z - 6yz - 4 = 0$

$\therefore$ The required enveloping cone is,

$$4x^2 + 3y^2 - 5z^2 - 8x + 16z - 6yz - 4 = 0.$$

Q7. **Find the equation of the enveloping cone of the sphere $x^2 + y^2 + z^2 = 169$ with the vertex at point A(5, 10, 15).**

Answer : **(OU) June/July-22, Q12**

Given,

Equation of sphere is,

$$S \equiv x^2 + y^2 + z^2 - 169$$

$S_1 = 181$

And vertex is $A(5, 10, 15)$

The equation of the enveloping cone is given by,

$$SS_1 = T^2$$

$$\Rightarrow (x^2 + y^2 + z^2 - a^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2) = (\alpha x + \beta y + \gamma z - a^2)^2 \quad \text{... (1)}$$

Here, $a^2 = 169$, $\alpha = 5$, $\beta = 10$, $\gamma = 15$

Substituting the corresponding values in equation (1), $(a-b)^2$

$$(x^2 + y^2 + z^2 - 169)(5^2 + 10^2 + 15^2 - 169) = (5x + 10y + 15z - 169)^2$$

$$\Rightarrow (x^2 + y^2 + z^2 - 169)(181) = (5x)^2 + (10y)^2 + (15z)^2 + 2(5x)(10y) + 2(10y)(15z) + 2(15z)(5x) + (169)^2 - 2(169)(5x + 10y + 15z)$$

$$\Rightarrow 181x^2 + 181y^2 + 181z^2 - 30589 = 25x^2 + 100y^2 + 225z^2 + 100xy + 300yz + 150xz + 28561 - 1690x - 3380y - 5070z$$

$$\Rightarrow 156x^2 + 81y^2 - 44z^2 - 100xy - 300yz - 150xz + 1690x + 3380y + 5070z - 59150 = 0$$

Which is the required equation.

now that general equation of a cone passing through 3 axes is

$fyz+gzx+hxy=0$ — (1)

proof.

general eqn of cone

$\boxed{ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0}$ (1)

The direction cosines
$(1,0,0)$ $(0,1,0)$ $(0\,0\,1)$
will be generators of cone pass through 3 axes
i.e. origin → vertex

eqn (1) pass $(1,0,0)$

$a(1)^2+0=0$

$a=0$

eqn (1) pass $(0,1,0)$

$b(1)^2+0=0$

$b=0$

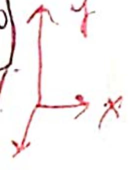
eqn (1) pass $(0,0,1)$

$c(1)^2=0 \Rightarrow c=0$

$a=b=c=0$ in eqn (1)

$\boxed{fyz+gzx+hxy=0}$

direction cosines



x

cos angles with co-ordinate axes

$l=\cos\alpha$

$m=\cos\beta$

$n=\cos\gamma$

⇒ x-axis

$y=90$ $z=90$

direction cosines of x-axes

$l=\cos 0=1$

$m=\cos 90=0$

$n=\cos 90=0$

direction cosines of axes

$=(1,0,0)$

pic

Q9. Show that the general equation of a cone which passes through the three axes is $fyz + gzx + hxy = 0$.

(KU) Dec.-18, Q1(a)

OR

Show that the general equation to a cone which passes through the three coordinate axes is $fyz + gzx + hxy = 0$ where f, g, h are parameters.

[(MGU) June-22, Q2 | (OU) Nov./Dec.-18, Q4]

Answer :

The general equation of cone with origin as its vertex is,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \dots (1)$$

Let the co-ordinate axes be its generators

Then the direction cosines of generators are $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$ respectively.

Key

Since $(1, 0, 0)$ passes through equation (1),

$a(1)^2 = 0$

$\Rightarrow a = 0$

easy

Since $(0, 1, 0)$ passes through equation (1),

$b(1)^2 = 0$

do it

$\Rightarrow b = 0$

Since $(0, 0, 1)$ passes through equation (1),

$c(1)^2 = 0$

$\Rightarrow c = 0$

$\therefore a = 0, b = 0, c = 0$

Substituting the above values in equation (1),

$0 + 0 + 0 + 2fyz + 2gzx + 2hxy = 0$

$\Rightarrow 2(fyz + gzx + hxy) = 0$

$\Rightarrow fyz + gzx + hxy = 0$

$\therefore$ The general equation which passes through three axes is, $fyz + gzx + hxy = 0$.

Q10. Find the equation to the cone which passes through the three coordinate axes as the two lines ...

(4) Show that general equation of cone which touches three co-ordinate planes is

Imp $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$ f, g, h parameters.

PU15 Co-or Touch

as we know general eqn of cone through co-ordinate ~~plane~~ axes is

$$fyz + gzx + hxy = 0 \quad \text{--- (1)}.$$

$$2fyz + 2gzx + 2hxy = 0 . \times 2 = 0$$

And reciprocal eqn of cone through eqn (1) is

is $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

Comparing $fyz + gzx + hxy = 0$ with
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ we get

$a=0$ $b=0$ $c=0$ $2f=f$ $2g=g$ $2h=h$

$f = \frac{f}{2}$ $g = \frac{g}{2}$ $h = \frac{h}{2}$

$A = bc - f^2 \Rightarrow A = 0 - \left(\frac{f}{2}\right)^2 \Rightarrow A = \frac{-f^2}{4}$

$B = ca - g^2 \Rightarrow B = 0 - \left(\frac{g}{2}\right)^2 \Rightarrow B = -g^2/4$

$C = ab - h^2 \Rightarrow C = 0 - \left(\frac{h}{2}\right)^2 \Rightarrow C = -h^2/4$

$F = gh - af \Rightarrow F = \left(\frac{g}{2}\right)\left(\frac{h}{2}\right) - 0 \Rightarrow F = \frac{gh}{4}$

$G = \cancel{hf} - \cancel{bg} \Rightarrow G = \frac{hf}{4} - 0 \Rightarrow G = \frac{hf}{4}$

$H = fg - ch \Rightarrow H = \frac{f}{2}\frac{g}{2} - 0 \Rightarrow H = fg/4$

$G = hf - bg$

Substituting values in
$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy$

we get,

$\frac{-f^2}{4}x^2 - \frac{g^2}{4}y^2 - \frac{h^2}{4}z^2 + 2\frac{gh}{4}yz + 2\frac{hf}{4}zx + 2\frac{fg}{4}xy = 0$

$\frac{-1}{4}\left[f^2x^2 - g^2y^2 - h^2z^2 - 2ghyz - 2hfzx - 2fgxy\right] = 0$

add & sub
Imp

$f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy + \underbrace{2fgxy - 2fgxy}_{=0}$

$f^2x^2 + g^2y^2 + h^2z^2 + 2fgxy - 2ghyz - 2hfzx - 4fgxy = 0$

$(fx + gy - hz)^2 = 4fgxy \Rightarrow fx + gy - hx = \pm\sqrt{4fgxy}$

$fx + gy - hz = \pm 2\sqrt{fgxy}$

$(A + B - C)^2 = A^2 + B^2 + C^2 + 2AB - 2BC - 2CA$

$fx + gy \pm 2\sqrt{fgxy} = hz$

$(\sqrt{fx} \pm \sqrt{gy})^2 = hz$

$(\sqrt{fx})^2 + (\sqrt{gy})^2 \pm 2\sqrt{fx}\sqrt{gy} = fx + gy \pm 2\sqrt{fgxy}$

$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

Q13. **Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal.**

OR

Show that the reciprocal cone of $ax^2 + by^2 + cz^2 = 0$ is the cone $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$.

(OU) July-21, Q4

Answer :

Piece of cone doubt

Given,

Equation of cone is,

$$ax^2 + by^2 + cz^2 = 0 \quad \dots (1)$$

Let the reciprocal cone be,

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \dots (2)$$

Comparing equation (1) with

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

$$a = a, b = b, c = c, f = g = h = 0$$

$$A = bc - f^2 = bc - 0 = bc$$

$$B = ca - g^2 = ca - 0 = ca$$

$$C = ab - h^2 = ab - 0 = ab$$

$$F = gh - af = 0$$

$$G = hf - bg = 0$$

$$H = fg - ch = 0$$

$$\therefore \quad A = bc, B = ca, C = ab, F = G = H = 0$$

Substituting the corresponding values in equation (2),

$$bcx^2 + cay^2 + abz^2 + 0 + 0 + 0 = 0$$

$$\Rightarrow \quad bcx^2 + acy^2 + abz^2 = 0$$

$$\Rightarrow \quad \frac{bcx^2 + acy^2 + abz^2}{abc} = 0$$

$$\Rightarrow \quad \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$$

$\therefore \quad ax^2 + by^2 + cz^2 = 0$ and $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ are reciprocal.

Q14. Find the equation of the cone reciprocal to the cone $5x^2 + 9y^2 + 11z^2 = 0$.

(OU) Nov./Dec.-18,

Answer :

Given,

Equation of cone is, gener

$$5x^2 + 9y^2 + 11z^2 = 0 \quad \text{...(1)}$$

Let the equation of reciprocal cone be,

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \text{...(2)}$$

Comparing equation (1) i.e., $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$,

$a = 5, \quad b = 9, \quad c = 11, \quad f = g = h = 0$

$A = bc - f^2 = (9)(11) - 0 = 99$

$B = ca - g^2 = (11)(5) - 0 = 55$

$C = ab - h^2 = (5)(9) - 0 = 45$

$F = gh - af = 0$

$G = hf - bg = 0$

$H = fg - ch = 0$

$\therefore \quad A = 99, B = 55, C = 45, F = G = H = 0.$

Substituting the corresponding values in equation (2),

$$99x^2 + 55y^2 + 45z^2 + 0 + 0 + 0 = 0$$

$$\Rightarrow \quad 99x^2 + 55y^2 + 45z^2 = 0$$

$$\Rightarrow \quad \frac{99x^2 + 55y^2 + 45z^2}{495} = \frac{0}{495}$$

$$\Rightarrow \quad \frac{x^2}{5} + \frac{y^2}{9} + \frac{z^2}{11} = 0$$

If f, g, h are zero ⇒ The equation has right i.e. expected Reciprocals

See once

$\therefore$ The equation of the cone reciprocal to the cone $5x^2 + 9y^2 + 11z^2 = 0$ is, $\frac{x^2}{5} + \frac{y^2}{9} + \frac{z^2}{11} = 0$.

Q15. Show that $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is the line of intersection of the tangent planes to the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ along the lines in which it cut by the plane $x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) = 0$

Answer :

do it

Given equation of cone is,

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{...(1)}$$

Equation of line is,

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{...(2)}$$

Equation of tangent plane at any point (α, β, γ) of equation (1) is,

$$x(a\alpha + h\beta + \gamma\gamma) + y(h\alpha + b\beta + f\gamma) + z(g\alpha + f\beta + c\gamma) = 0$$

$\Rightarrow \quad l(a\alpha + h\beta + g\gamma) + m(h\alpha + b\beta + f\gamma) + n(g\alpha + f\beta + c\gamma) = 0 \quad$ [$\because$ From equation (2)]

$\Leftrightarrow \quad \alpha(al + hm + gn) + \beta(hl + bm + fn) + \gamma(gl + fm + cn) = 0$

Since, the point (α, β, γ) lies on above plane equation,

i.e., $x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) = 0$

Which is the required plane equation.

4) prove that the tangent planes to the cone $x^2-y^2-2z^2-3yz+4zx-5xy=0$ are perpendicular to the generators of the cone $17x^2+8y^2+29z^2+28yz-46zx-16xy=0$

Answer: Given equations are

$$x^2-y^2+2z^2-3yz+4zx-5xy=0 \quad \dots (1)$$

$$17x^2+8y^2+29z^2+28yz-46zx-16xy=0 \quad (2)$$

The tangent planes to cone equation (1) are said to be prependicular to the generators of cone equation (2) if equations (1) are reciprocal

Let the reciprocal cone of equation (1) be

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$$

comparing equation (1) with $ax^2+by^2+cz^2+2fyz+2hxy=0$,

$$a=1,\ b=-1,\ c=2,\ f=\frac{-3}{2}\ g=2\ h=\frac{-5}{2}$$

$$A=bc-f^2=-1(2)-\left(\frac{-3}{2}\right)^2=-\frac{17}{4}$$

$$c=ab-h^2=1(-1)-\left(\frac{-5}{2}\right)=-\frac{29}{4}$$

$$F=gh-af=2\left(\frac{-5}{2}\right)-(1)\left(\frac{-3}{2}\right)=-\frac{7}{2}$$

$$F=gh-af=2\left(\frac{-5}{2}\right)-(1)\left(\frac{-3}{2}\right)=-\frac{7}{2}$$

$$G=hf-bg=\frac{-5}{2}\left(\frac{-3}{2}\right)-(-1)(2)=\frac{23}{4}$$

$$H=fg-ch=\frac{-3}{2}(2)-2\left(\frac{-5}{2}\right)=2$$

substituting the corresponding values in equation (3)

$$\frac{-17}{4}x^2-2y^2-\frac{29}{4}z^2+2\left(\frac{-7}{2}\right)yz+2\left(\frac{23}{4}\right)zx+2(2)xy=0$$

$$\Rightarrow \frac{-17}{4}x^2-2y^2-\frac{29}{4}z^2-7yz+\frac{46}{4}zx+4xy=0$$

$$\Rightarrow -17x^2-8y^2-29z^2-28yz+46zx+16xy=0$$

$$\Leftrightarrow 17x^2+8y^2+29z^2+28yz-46zx-16xy=0$$

The above equation is same as equation (2)

$\therefore$ Equations (1) and (2) are reciprocal

Hence the tangents to $x^2-y^2+2z^2-3yz+4zx-5xy=0$ are perpendicular to the generators of the cone $17x^2+8y^2+29z^2+28yz-46zx-16xy=0$

Find reciprocal cone of $4x^2+5y^2+7z^2=0$ — (1)

Here $a=4$ $b=5$, $c=7$ $f=g=h=0$

eqn of reciprocal cone is

$$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0 \quad (2)$$

$A=bc-f^2=5(7)-0=35$

$B=ca-g^2=7(4)-0=28$

$C=ab-h^2=4(5)-0=20$

$F=gh-af=0$

$G=hf-bg=0$

$H=fg-ch=0$

Sub in (2)

$35x^2+28y^2+20z^2=0$

divide by 140 we get

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{7}=0$$

$$4(l^2+m^2+n^2)+2(l-2m+3n)(l-m+n)-4(l-m+n)^2=0$$

$(l+(-m)+n)^2$

$$4l^2+4m^2+4n^2+2[l^2-lm+ln-2ml+2m^2-2mn+3nl-3nm+3n^2]$$

$2ln$

$$-4[l^2+m^2+n^2-\cancel{2lm}-2mn+\cancel{2ln}]=0$$

$-2lm$

$$4l^2+4m^2+4n^2+2l^2-2lm+2ln-4ml+4m^2-4mn$$

$$6nl-6nm+6n^2-4l^2-4m^2-4n^2+8mn=0$$

$+8lm-8ln$

$$2l^2+4m^2+6n^2+2ln+2mn=0$$

$$2[l^2+2m^2+3n^2+ln+mn]=0$$

$$x^2+2y^2+3z^2+xy+yz=0$$

eqn of cone

find the ... vertex is $(1,2,3)$ and guiding curve is the circle $x^2+y^2+z^2=4$, $x+y+z=1$

Answers Given,

vertex is $(1,2,3)$

And guiding curve is.

$x^2+y^2+z^2=4$

$x+y+z=1$

Then the equation of line though $(1,2,3)$ is,

$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=r$$

Equating the terms.

$\frac{x-1}{l}=r \Rightarrow y=mr+1$

$\frac{y-2}{m}=r \Rightarrow y=mr+2$

$\frac{z-3}{n}=r \Rightarrow z=nr+3$

$\therefore (x,y,z)=(lr+1, mr+2, nr+3)$ lies on equations (1) and (2)

i.e. $(lr+1)^2+(mr+2)^2+(nr+3)^2=4$

$\Rightarrow l^2r^2+1+2lr+m^2r^2+4+4mr+n^2r^2+9+6nr-4=0$

$\Rightarrow (l^2+m^2+n^2)r^2+(2l+4m\,6n)r+10=0$

Also, $lr+1+mr+2+nr+3-1=0$

$\Rightarrow (l+m+n)r+5=0$

$\Rightarrow r=\frac{-5}{r+m+n}$

substituting equation (5) in equation (4)

$$(l^2+m^2+n^2)\left[\frac{-5}{l+m+n}\right]^2+(2l+4m+6n)\left[\frac{-5}{l+m+n}\right]+10=0$$

$$= \frac{(2l+4m+6n)}{l+m+n} + 10$$

$$+6n)(l+m+n)+10$$

Prove that $2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0$ represents a cone with vertex $(2,2,1)$

Sol:- given eqution is

$$2x^2+2y^2+7z^2-10yz-10zx+2x+2y+26z-17=0 \quad -(1)$$

vertex $(2,2,1)$

Homogenizing eqn (1).

$$F(x,y,z,t) = 2x^2+2y^2+7z^2-10yz-10zx+2xt+2yt+26zt-17t^2 \quad -(2)$$

for $t=1$

partially differentiating eqn (2) with res to x, y, z and t respectively

$F_x = 0$.

$4x-10z+2=0 \quad -(3)$

$F_y = 0$

$4y-10z+2=0 \quad -(4)$

$F_z = 0$

$14z-10y-10x+26=0$

$10x-10y+14z+26=0 \quad -(5)$

$F_t = 0$

$2x+2y+26z-34=0 \quad -(6)$

solving (3) & (4)

$4x-10z+2=0$

$4y-10z+2=0$

$\underline{- \quad + \quad -}$

$4x-4y=0$

$4x=4y \Rightarrow x=y$.

Sub $z=1$ in eqn (4)

$\Rightarrow 4y-10(1)+2=0 \Rightarrow 4y=8 \Rightarrow \boxed{y=2}$ $\boxed{x=2}$

~~solving~~ Substituting $x=y$ in eqn (6)

$2y+2(y)+26z-34=0$.

$4y+26z-34=0 \quad -(7)$

Solving (7) & (4)

$4y+26z-34=0$

$4y-10z+2=0$

$\underline{- \quad + \quad -}$

$36z-36=0 \Rightarrow 36z=36$

$\boxed{z=1}$

Substituting x, y, z in eq (5)

$10x + 10y - 14z - 26 = 0$

$10(2) + 10(2) - 14(1) - 26 = 0$

$20 + 20 - 14 - 26 \Rightarrow 40 - 40 = 0$

$\therefore$ eqns (3), (4), (5), (6) are consistent

at $(2,2,1)$ as vertex.

$\therefore$ given eqution represents a cone with vertex $(2,2,1)$.

(7) Find equations to the lines in which the plane $2x+y-z=0$ cuts the cone

$4x^2 - y^2 + 3z^2 = 0$

Given plane is $2x + y - z = 0$ — (1)

eqn of cone is $4x^2 - y^2 + 3z^2 = 0$ — (2)

let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ — (3) be required line eqn.

from eqn (1) & (2) $2l + m - n = 0 \Rightarrow n = 2l + m$

$4l^2 - m^2 + 3n^2 = 0$.

$4l^2 - m^2 + 3(2l + m)^2 = 0$

$4l^2 - m^2 + 3(4l^2 + m^2 + 4lm) = 0$

$4l^2 - m^2 + 12l^2 + 3m^2 + 12lm = 0$

$16l^2 + 2m^2 + 12lm = 0$

$2(8l^2 + m^2 + 6lm) = 0 \Rightarrow 2 \neq 0$

$8l^2 + 6lm + m^2 = 0$

$8\frac{l^2}{m^2} + \frac{6lm}{m^2} + \frac{m^2}{m^2} = 0$

$8\left(\frac{l}{m}\right)^2 + 6\left(\frac{l}{m}\right) + 1 = 0$.

it is in quadratic eqn form

$$8\left(\frac{l}{m}\right)^2+6\left(\frac{l}{m}\right)+1=0.$$

$$\frac{l}{m}=\frac{-6\pm\sqrt{6^2-4(8)(1)}}{2(8)}=\frac{-6\pm\sqrt{36-32}}{16}=\frac{-6\pm 2}{16}$$

$$\frac{l}{m}=\frac{-6+2}{16};\quad \frac{l}{m}=\frac{-6-2}{16}$$

$$\frac{l}{m}=\frac{-1}{4};\quad \frac{l}{m}=\frac{-1}{2}$$

from eqn $n=2l+m$

$$\frac{n}{m}=2\frac{l}{m}+1$$

from $\frac{l}{m}=\frac{-1}{4} \Rightarrow \frac{n}{m}=2\left(\frac{-1}{4}\right)+1$

$$\frac{n}{m}=\frac{-1}{2}+1=\frac{-1+2}{2}=\frac{1}{2}$$

$$\frac{n}{m}=\frac{1}{2}.$$

$$\frac{l}{-1}=\frac{m}{4}=\frac{n}{2}$$

for $\frac{l}{m}=\frac{-1}{2} \Rightarrow \frac{n}{m}=2\left(\frac{-1}{2}\right)+1 \Rightarrow \frac{n}{m}=0$

$$\frac{l}{-1}=\frac{m}{2};\ n=0$$

$\therefore$ from eqn ③ $l=-1, m=4, n=2 \Rightarrow \frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$

$l=-1,\ m=2,\ n=0 \Rightarrow \frac{x}{-1}=\frac{y}{2};\ z=0$

$\therefore$ lines required are

$$\frac{x}{-1}=\frac{y}{4}=\frac{z}{2} \quad \text{and} \quad \frac{x}{-1}=\frac{y}{2};\ z=0$$

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9) PT angle b/w. $x+y+z=0$, $ayz+bzx+cxy=0$

is $\frac{\pi}{2}$ if $a+b+c=0$ and

$\frac{\pi}{3}$ if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

given $x+y+z=0$ — (1)

cone eqn $ayz+bzx+cxy=0$ — (2)

let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — (3) are required eqn of line

from (1) $l+m+n=0 \Rightarrow m=-l-n$

from (2) $amn+bnl+cnm=0$

$a(-l-n)n+bnl+cn(-l-n)=0$

$-aln-an^2+bnl-cnl-cn^2=0$

con eqn $ayz+bzx+cxy=0$

let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ (3) are required eqn

$\Rightarrow$ from (1) $l+m+n=0 \Rightarrow n=-l-m$

from (2) $amn+bnl+cnm=0$.

$am(-l-m)+b(-l-m)l+c(-l-m)m=0$

$-aml-am^2-bl^2-bml-cml-cm^2=0$

$-(a+b+c)ml-(a+c)m^2-bl^2=0$

if $a+b+c=0 \Rightarrow$ above eqn.

$a+c=-b$

$-(0)ml-(-b)m^2-bl^2=0$

$0+bm^2=bl^2$

$\Rightarrow \frac{b}{b}=\frac{l^2}{m^2} \Rightarrow \frac{l^2}{m^2}=\frac{1}{1} \Rightarrow \frac{l}{m}=\frac{\pm 1}{1}$

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$a^2 - a - 6 = 0 \Rightarrow a^2 - 3a + 2a - 6 = 0$

$a(a-3) + 2(a-3) = 0$

$(a-3)\cdot(a+2) = 0 \Rightarrow a = 3,\ a = -2.$

$a = \frac{m}{n} = -2$ or $\frac{m}{n} = 3.$

from eqn 4

$l = \frac{6n - 7m}{10}$

$\frac{10l}{n} = 6 - \frac{7m}{n}$

$10\frac{l}{n} = 6 - 7(-2)$

$10\frac{l}{n} = 6 + 14 = 20.$

$\frac{l}{n} = \frac{20}{10} = \frac{2}{1}$

from eqn $\frac{l}{2} = \frac{m}{-2} = \frac{n}{1}$

$\frac{10l}{n} = 6 - \frac{7m}{n}.$

$\frac{10l}{n} = 6 - 7(3)$

$10\frac{l}{n} = 6 - 21 = -15$

$\frac{l}{n} = \frac{-15}{10} = \frac{-3}{2}$

and $\frac{l}{-\frac{3}{2}} = \frac{m}{3} = \frac{n}{1}$

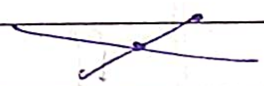
$\therefore$ from eqn (3) required lines are

$\frac{x}{2} = \frac{y}{-2} = \frac{z}{1}$ and $\frac{x}{-\frac{3}{2}} = \frac{y}{3} = \frac{z}{1}$

direction ratios of lines are

$(2, -2, 1)$ and $\left(\frac{-3}{2}, 3, 1\right)$

If θ is angle b/w the lines of intersection



$\Rightarrow \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

$\cos\theta = \frac{2\left(-\frac{3}{2}\right) + (-2)3 + 1(1)}{\sqrt{2^2 + (-2)^2 + 1^2}\sqrt{\left(\frac{-3}{2}\right)^2 + 3^2 + 1}}$

$\cos\theta = \frac{-3 + 1 - 6}{\sqrt{4+4+1}\sqrt{\frac{9}{4} + 9 + 1}} = \frac{8}{\sqrt{9}\sqrt{\frac{49}{4}}} = \frac{8}{3\cdot\frac{7}{2}} = \frac{8 \times 2}{21}$

$\cos\theta = \frac{16}{21},\ \Rightarrow \boxed{\theta = \cos^{-1}\frac{16}{21}}$

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(9) PT angle b/w. $x+y+z=0$, $ayz+bzx+cxy=0$ is $\frac{\pi}{2}$ if $a+b+c=0$ and $\frac{\pi}{3}$ if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

given $x+y+z=0$ — (1)

cone eqn $ayz+bzx+cxy=0$ — (2)

let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — (3) are required eqn of line

~~from (1) $l+m+n=0 \Rightarrow m=-l-n$~~

~~from (2) $amn+bnl+cnm=0$~~

~~$a(-l-n)n+bnl+cn(-l-n)=0$~~

~~$-aln-an^2+bnl-cnl-cn^2=0$~~

Con eqn $ayz+bzx+cxy=0$

let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — (3) are required eqn

$\Rightarrow$ from (1) $l+m+n=0 \Rightarrow n=-l-m$

from (2) $amn+bnl+clm=0$.

$am(-l-m)+b(-l-m)l+c(-l-m)m=0$

$-aml-am^2-bl^2-bml-cml-cm^2=0$

$-(a+b+c)ml-(a+c)m^2-bl^2=0$.

If $a+b+c=0 \Rightarrow$ above eqn.

$a+c=-b$

$-(0)ml-(-b)m^2-bl^2=0$

$0+bm^2=bl^2$

$\Rightarrow \frac{b}{b}=\frac{l^2}{m^2} \Rightarrow \frac{l^2}{m^2}=\frac{1}{1} \Rightarrow \frac{l}{m}=\frac{\pm 1}{1}$

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from ① $l = -m - n$

$$\frac{l}{m} = -1 - \frac{n}{m}$$

$\frac{l}{m} = \frac{+1}{1} \Rightarrow \frac{1}{1} = -1 - \frac{n}{m} \Rightarrow 1+1 = -\frac{n}{m} \Rightarrow \frac{-2}{1} = \frac{n}{m}$

$\frac{l}{m} = \frac{-1}{1} \Rightarrow \frac{-1}{+1} = -1 - \frac{n}{m} \Rightarrow -1+1 = \frac{-n}{m} \Rightarrow \frac{0}{1} = \frac{n}{m}$

$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$ and $\frac{l}{-1} = \frac{m}{1} = \frac{n}{0}$

from eqn ③ $\frac{x}{1} = \frac{y}{1} = \frac{z}{-2}$ & $\frac{x}{-1} = \frac{y}{1} = \frac{z}{0}$

$\Rightarrow (1, 1, -2) \quad (-1, 1, 0)$

let θ be angle $\Rightarrow \cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$

$$\cos\theta = \frac{1(-1) + 1(1) + (-2)(0)}{\sqrt{1^2+1^2+(-2)^2}\sqrt{(-1)^2+1^2+0^2}} = \frac{-1+1+0}{\sqrt{1+1+4}\sqrt{1+1}}$$

$$\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}.$$

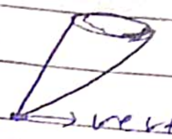
$\therefore$ when $a+b+c=0$; $\Rightarrow$ angle is $\frac{\pi}{2}$.

Formulas

Unit-2 Cone :-

(1) what is cone :-

A surface generated by a straight line passing through a fixed point and intersecting a curve or touching a given surface is known as cone.



→ general equation of cone :-

eqn of cone with vertex (α, β, γ) is

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 + 2f(z-\gamma)(y-\beta) + 2g(x-\alpha)(z-\gamma) + 2h(x-\alpha)(y-\beta) = 0$$

→ eqn of cone passing through 3 axes :-

$$fyz + gzx + hxy = 0$$ & f, g, h parameters.

→ Enveloping cone :- Cone formed by tangent lines to a surface from a point or vertex is called Enveloping cone

→ eqn of line passing through vertex (α, β, γ)

$$\cancel{\frac{x-\alpha}{l}} \quad \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

vertex, base x or y or z ఇస్తే cone ఉంటే
ఆ line eqn లో ఇచ్చిన x/y/z value sub
చేయాలి.

⇒ దాని remaining 2 seperate equate చేసి
x, y, మరియు z first ఇచ్చిన value ల

⇒ ఈ x, y, z ని base eqn లో sub → cone eqn
eliminate చేస్తే

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$(A+B+C)^2 = A^2+B^2+C^2+2AB+2BC+2CA$

$(A+B-C)^2 = A^2+B^2+C^2+2AB-2AC-2BC$

Sphere equation S & vertex (point) $P(\alpha,\beta,\gamma)$

enveloping cone

is $SS_1 = T^2$ is required enveloping cone

S_1 = S on point P/vertex substitute value

$T_1 = \alpha x+\beta y+\gamma z+u(x+\alpha)+v(y+\beta)+w(z+\gamma)+d$

$u, v, w \to$ sphere

Reciprocal of cone

~~at condition~~ $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=0$

$ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ — (1)

is eqn of cone $\Rightarrow$ then

$Ax^2+By^2+Cz^2$

Reciprocal of cone :-

If $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ — (1)

eqn of cone then

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

represents Reciprocal of eqn (1)

where

$A = bc-f^2$

$B = ca-g^2$

$C = ab-h^2$

$F = gh-af$

$G = hf-bg$

$H = fg-ch$

$A \to f$, $b\ c$

$B \to g$, $a\ c$

$C \to h$, $a\ b$

F: a, gh

G: b, fh

H: c, fg